3D wave field scattered by thin elastic bodies buried in an elastic medium using the Traction Boundary Element Method

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Abstract

A combined formulation based on the Traction Boundary Element Method (TBEM) and the Boundary Element Method (BEM) is proposed to model three-dimensional wave scattering in solid media, which contain thin elastic bodies whose geometry remains the same along one direction. Although the classical BEM models degenerate in the presence of thin heterogeneities, this combined formulation, in which one side of the body is discretized with the BEM and the other side with the TBEM formulation, is able to model the wave field in the vicinity of these heterogeneities. Arbitrary-shaped and oriented elastic thin bodies, even with no thickness, may be modelled using this formulation. Analytical integrations are performed to evaluate the hypersingular integrals. The proposed model is applied to two different thin heterogeneities buried in an elastic unbounded medium.

Keywords: wave propagation, elastic scattering, thin elastic inclusion, BEM, TBEM, two-and-a-half-dimensional problem.

1 Introduction

The problem of modeling the scattering of elastic waves by an elastic inclusion, a cavity or a crack has been addressed in different fields of research related to the remote detection, location and identification of heterogeneities, delaminations or anomalies. A complete understanding of how waves propagate from a generic source to a receiver in a homogeneous elastic medium and in the vicinity of irregularities, the so-called direct problem analysis, is required for the correct interpretation of many of those testing techniques [1–4].
Since analytical solutions have only been derived for objects with simple geometry [5], several numerical methods have been proposed over the years to study the detection of defects and heterogeneities by elastic wave scattering. Among these numerical methods, the Boundary Element Method (BEM) seems quite suitable for wave propagation modeling in homogeneous unbounded systems [6,7]. However, when the heterogeneity is very thin, such as a crack or almost imperceptible defect, the conventional direct BEM degenerates.

The Indirect Boundary Element formulation [8], the Traction Boundary Integral Equation Method [9] and the Traction Boundary Element Method (TBEM) [10] are among the numerical methods that solve the thin-body difficulty. These formulations require the solution of hypersingular integrals, and different techniques have been proposed to cope with this complexity [11,12]. Prosper & Kausel [10] proposed an indirect approach for the analytical evaluation of integrals with hypersingular kernels for plane-strain cases in a two-dimensional (2D) medium. Another boundary element formulation, mainly adopted for the analysis of crack problems, is referred to as the Dual Boundary Element Method. In this technique, a single region formulation is achieved by combining the displacement boundary integral equation discretizing one of the crack surfaces and the traction boundary integral equation used to discretize the opposite crack surface [13].

This paper describes a combined formulation based on the TBEM and the BEM that is used to model the three-dimensional (3D) wave scattering in solid media, which contain thin elastic bodies whose geometry remains the same along one direction. Firstly, the 3D problem formulation and the boundary elements formulations are presented. Afterwards, the implementation of this model is successfully verified for the case of an elastic circular inclusion by using analytical solutions, already known for simple geometry cases.

An application of the proposed model is presented for a softer and a harder thin heterogeneity buried in an elastic unbounded medium. As the model is initially formulated in the frequency domain, time results are obtained by applying inverse fast Fourier transformations to the frequency responses. Time results are presented for the displacement fields and both the wave pattern evolution and the perturbation due to the presence of the thin bodies in the elastic medium can be clearly observed.

2 Problem formulation

An infinite 2D elastic inclusion is buried in an unbounded homogeneous isotropic elastic medium, with no intrinsic attenuation. The exterior medium (with density $\rho_1$) allows shear wave and compressional wave velocities of $\beta_1$ and $\alpha_1$, respectively. The inclusion is aligned along the $z$ axis and perfectly filled with a different solid material with density $\rho_2$, shear wave and compressional wave velocities of $\beta_2$ and $\alpha_2$, respectively. The host medium is excited by a harmonic dilatational 3D point source, placed at $O (x, y, 0)$, which
oscillates with frequency $\omega$ and emits an incident field that can be given by a dilatational potential $\phi$, 

$$\phi_{\text{inc}} = A e^{i\omega t} \frac{e^{-i\sqrt{k_x^2 + k_y^2 + k_z^2}}}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}},$$

with 'inc' referring to the incident field, $A$ the wave amplitude and $i = \sqrt{-1}$.

As the geometry of this problem does not change along the $z$ direction, applying a Fourier transformation along that direction allows the 3D solution to be expressed as a summation of 2D problems, each one solved for a specific spatial wavenumber, $k_{\alpha}$,

$$\phi_{\text{inc}}(\omega, x, y, z) = \frac{2\pi}{L_{\text{ns}}} \sum_{m=1}^{\infty} \phi_{\text{inc}}(\omega, x, y, k_{zm}),$$

where $\phi_{\text{inc}}(\omega, x, y, k_{zm}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{\text{inc}}(\omega, x, y, k_{zm}) \exp(-i\omega t) \, dt$, $k_{zm}$ is the axial (longitudinal) wavenumber defined by $k_{zm} = \frac{2\pi}{L_{\text{ns}}}$, the effective wavenumbers are given by $k_{\alpha} = \sqrt{k_x^2 + k_y^2 + k_z^2}$, with $\text{Im}(k_{\alpha}) < 0$, $H_n(\cdot)$ are second Hankel functions of order $n$, and $L_{\text{ns}}$ is the distance between virtual point sources periodically placed along $z$. If the space between sources is kept large enough to prevent spatial contamination from the virtual sources, that summation converges and an approximation can be given by a finite sum of terms [8]. The pure 2D problem corresponds to $k_z = 0$.

Solutions in the time domain are obtained by applying an inverse Fourier transform to the frequency domain responses, where the dynamic excitation source is modelled as a Ricker pulse. To avoid the aliasing phenomena and to minimize the contribution of the periodic virtual sources, complex frequencies with an imaginary part of the form $\omega_z = \omega - i\eta$ (with $\eta = 0.7\Delta \omega$) are used [8].

3 Boundary integral formulations

A combined formulation based on the TBEM and the BEM, will be presented to model the 3D wave scattering in solid media which contain thin elastic bodies whose geometry remains the same along one direction. Prior to the combined formulation, both boundary elements methods are presented.

3.1 Boundary element formulation

An elastic body bounded by a surface $S$ is buried in a homogeneous infinite elastic medium, which is illuminated by a spatially sinusoidal harmonic line load located in the exterior solid medium at $x_i$, with axial wavenumber $k_z$. The wave propagation in this system is governed by boundary integral equations, which can be derived by applying the reciprocity theorem [6], leading firstly to:
along boundary surface $S$, in the host elastic domain

$$c_y u_i(x_0, \omega) = \int_S t_j(x, n_\omega, \omega) G_{ij}(x, x_0, \omega) \, ds$$

$$- \int_S u_j(x, \omega) H_{ij}(x, n_\omega, x_0, \omega) \, ds + u_i^{\text{inc}}(x, x_0, \omega),$$

where $i, j = 1, 2$ correspond respectively to the normal and tangential directions relative to the inclusion surface, and $i, j = 3$ to the $z$ direction. $H_{ij}(x, n_\omega, x_0, \omega)$ and $G_{ij}(x, x_0, \omega)$ represent the fundamental traction and displacement solutions (Green’s functions) in direction $j$ on boundary $S$, at $x$, due to a unit point force applied in direction $i$ at collocation point, $x_0$. $u_j(x, \omega)$ and $t_j(x, n_\omega, x_0, \omega)$ refer to the displacement and traction unknowns in direction $j$ at $x$. $u_i^{\text{inc}}(x, x_0, \omega)$ corresponds to the displacement incident field at the collocation point along direction $i$. The coefficient $c_y$ is equal to $\delta_y/2$, with $\delta_y$ the Kronecker delta, for a smooth boundary. The unit outward normal at the boundary at $x$ is defined by the vector $n_\omega = (\cos \theta_\omega, \sin \theta_\omega)$.

The Green’s functions for loads and displacements in the $x$, $y$ and $z$ directions in an elastic medium have been derived and presented by Tadeu and Kausel [14]. The derivatives of these Green’s functions make it possible to find the following tractions along the $x$, $y$ and $z$ directions, in the solid medium,

$$H_{rx} = 2\mu \left[ \frac{\alpha^2}{2\beta^2} \frac{\partial G_{rx}}{\partial x} + \left( \frac{\alpha^2}{2\beta^2} - 1 \right) \left( \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rz}}{\partial z} \right) \right] \cos \theta_n + \mu \left[ \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rx}}{\partial x} \right] \sin \theta_n$$

$$H_{ry} = 2\mu \left[ \frac{\alpha^2}{2\beta^2} \frac{\partial G_{rx}}{\partial x} + \left( \frac{\alpha^2}{2\beta^2} - 1 \right) \left( \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rz}}{\partial z} \right) \right] \sin \theta_n + \mu \left[ \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rx}}{\partial x} \right] \cos \theta_n$$

$$H_{rz} = \mu \left[ \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{rz}}{\partial z} \right] \cos \theta_n + \mu \left[ \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rz}}{\partial z} \right] \sin \theta_n$$

where $H_{rt} = H_{rt}(x, n_\omega, x_0, \omega)$, $G_{rt} = G_{rt}(x, x_0, \omega)$ and $r, t = x, y, z$. These expressions can be combined so as to give $H_{ij}(x, n_\omega, x_0, \omega)$ in the normal and tangential directions. In these equations, $\mu = \rho c^2$.

A similar boundary integral equation to that in eqn. (3) is defined along the boundary in the interior domain, where the elastic properties are those of the internal inclusion medium. The computation of these two boundary integral equations requires the discretization of both the boundary and boundary values. A system of linear equations is obtained by applying a virtual load to each node on the boundary in turn, and these can be solved to determine the nodal tractions and displacements. When the element to be integrated is the loaded one, the necessary integrations are performed analytically [15, 16], but when the element to be integrated is not the loaded element a numerical Gaussian quadrature scheme is applied.
3.2 Traction boundary element formulation

In the presence of a thin elastic inclusion, the previous BEM formulation degenerates. The TBEM can be derived [10] to cope with this limitation, leading to the following boundary integral equation:

\[
\begin{align*}
\sum_{j} c_{ij}(x_{i}, n_{i}, \omega) + a_{ij} u_{j}(x_{j}, \omega) & = \int_{S} t_{ij}(x, n_{i}, \omega) \mathcal{G}_{ij}(x, n_{i}, x_{j}, \omega) \, ds \\
& - \int_{S} u_{j}(x, \omega) \mathcal{H}_{ij}(x, n_{i}, x_{j}, \omega) \, ds + \bar{u}_{inc}(x_{i}, n_{i}, \omega),
\end{align*}
\]

where \( i, j = 1, 2 \) refer respectively to the normal and tangential directions relative to the inclusion surface, and \( i, j = 3 \) to the \( z \) direction. This equation can be understood as resulting from the application of dipoles (dynamic doublets) to the previous displacement boundary integral. Coefficients \( c_{ij} \) are defined as described above while \( a_{ij} \) are zero for piecewise straight boundary elements. Green’s functions \( \mathcal{G}_{ij}(x, n_{i}, x_{j}, \omega) \) and \( \mathcal{H}_{ij}(x, n_{i}, x_{j}, \omega) \) are given by the application of the traction operator to \( \mathcal{G}_{ij}(x, x_{j}, \omega) \) and \( \mathcal{H}_{ij}(x, n_{i}, x_{j}, \omega) \), which can be seen as the combination of the derivatives of the displacement boundary integral, in order to \( x, y \) and \( z \), so as to obtain stresses \( \mathcal{G}_{ij}(x, n_{i}, x_{j}, \omega) \) and \( \mathcal{H}_{ij}(x, n_{i}, x_{j}, \omega) \).

Along the boundary element, at \( x \), after performing the equilibrium of stresses, the following equations are expressed for \( x, y \) and \( z \) generated by loads also applied along the \( x, y \) and \( z \) directions:

\[
\begin{align*}
\mathcal{G}_{rx} & = 2 \mu \left[ \frac{\alpha^{2}}{2 \beta^{2}} \frac{\partial G_{rx}}{\partial x} + \left( \frac{\alpha^{2}}{2 \beta^{2}} - 1 \right) \left( \frac{\partial G_{ry}}{\partial y} + \frac{\partial G_{rz}}{\partial z} \right) \right] \cos \theta_{0} + \mu \left[ \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{ry}}{\partial y} \right] \sin \theta_{0} \\
\mathcal{G}_{ry} & = 2 \mu \left[ \frac{\alpha^{2}}{2 \beta^{2}} \left( \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{rz}}{\partial z} \right) + \frac{\alpha^{2}}{2 \beta^{2}} \frac{\partial G_{ry}}{\partial y} \right] \sin \theta_{0} + \mu \left[ \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{ry}}{\partial y} \right] \cos \theta_{0} \\
\mathcal{G}_{rz} & = \mu \left[ \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{rz}}{\partial z} \right] \cos \theta_{0} + \mu \left[ \frac{\partial G_{rx}}{\partial x} + \frac{\partial G_{ry}}{\partial y} \right] \sin \theta_{0} \\
\mathcal{H}_{rx} & = 2 \mu \left[ \frac{\alpha^{2}}{2 \beta^{2}} \frac{\partial H_{rx}}{\partial x} + \left( \frac{\alpha^{2}}{2 \beta^{2}} - 1 \right) \left( \frac{\partial H_{ry}}{\partial y} + \frac{\partial H_{rz}}{\partial z} \right) \right] \cos \theta_{0} + \mu \left[ \frac{\partial H_{rx}}{\partial x} + \frac{\partial H_{ry}}{\partial y} \right] \sin \theta_{0} \\
\mathcal{H}_{ry} & = 2 \mu \left[ \frac{\alpha^{2}}{2 \beta^{2}} \left( \frac{\partial H_{rx}}{\partial x} + \frac{\partial H_{rz}}{\partial z} \right) + \frac{\alpha^{2}}{2 \beta^{2}} \frac{\partial H_{ry}}{\partial y} \right] \sin \theta_{0} + \mu \left[ \frac{\partial H_{rx}}{\partial x} + \frac{\partial H_{ry}}{\partial y} \right] \cos \theta_{0} \\
\mathcal{H}_{rz} & = \mu \left[ \frac{\partial H_{rx}}{\partial x} + \frac{\partial H_{rz}}{\partial z} \right] \cos \theta_{0} + \mu \left[ \frac{\partial H_{rx}}{\partial x} + \frac{\partial H_{ry}}{\partial y} \right] \sin \theta_{0}
\end{align*}
\]

with \( n_{0} = (\cos \theta_{0}, \sin \theta_{0}) \) defining the unit outward normal at the collocation point, \( x_{0} \), \( G_{rt} = G_{rt}(x, n_{r}, x_{0}, \omega) \), \( G_{rt} = G_{rt}(x, x_{0}, \omega) \), \( H_{rt} = H_{rt}(x, n_{r}, x_{0}, \omega) \) and \( r, t = x, y, z \).

As with \( G_{rt} \) and \( H_{rt} \), the incident field component (referring to stresses) is evaluated by comparable expressions:
\[-u_{x}^{inc} = 2\mu \left[ \frac{\alpha^2}{2\beta^2} \frac{\partial u_{x}^{inc}}{\partial x} + \left( \frac{\alpha^2}{2\beta^2} - 1 \right) \left( \frac{\partial u_{y}^{inc}}{\partial y} + \frac{\partial u_{z}^{inc}}{\partial z} \right) \right] \cos \theta_0 + \mu \left[ \frac{\partial u_{y}^{inc}}{\partial x} + \frac{\partial u_{z}^{inc}}{\partial y} \right] \sin \theta_0 \]
\[-u_{y}^{inc} = 2\mu \left( \frac{\alpha^2}{2\beta^2} - 1 \right) \left( \frac{\partial u_{y}^{inc}}{\partial y} + \frac{\partial u_{z}^{inc}}{\partial z} \right) + \frac{\alpha^2}{2\beta^2} \frac{\partial u_{y}^{inc}}{\partial y} \right] \sin \theta_0 + \mu \left[ \frac{\partial u_{y}^{inc}}{\partial x} + \frac{\partial u_{z}^{inc}}{\partial y} \right] \cos \theta_0 \]
\[-u_{z}^{inc} = \mu \left[ \frac{\partial u_{z}^{inc}}{\partial x} + \frac{\partial u_{z}^{inc}}{\partial y} \right] \cos \theta_0 + \mu \left[ \frac{\partial u_{y}^{inc}}{\partial z} + \frac{\partial u_{x}^{inc}}{\partial y} \right] \sin \theta_0 \]

where \(-u_r^{inc} = u_r^{inc} (x_s, x_0, n_\delta, \omega), \quad u_r^{inc} = u_r^{inc} (x_s, x_0, \omega)\) and \(r = x, y, z\).

Fundamental solutions \(\tilde{G}_{ij}(x, n_\delta, x_0, \omega), \quad \tilde{H}_{ij}(x, n_\delta, x_0, \omega)\) and \(-u_i^{inc}(x_s, x_0, n_\delta, \omega)\) in the normal and tangential directions are obtained by combining the previous expressions.

As before, in the boundary element formulation, a similar boundary integral equation can be obtained along the boundary in the interior elastic domain. The solutions of the two traction boundary integral equations are evaluated, as before, by discretizing the boundary into \(N\) straight boundary elements, with one nodal point in the middle of each element. This procedure leads to a set of integrations, which are performed using a numerical Gaussian quadrature scheme, if the element to be integrated is not the loaded element. If the element to be integrated is the loaded one, hypersingular integrals are obtained which are evaluated through an indirect approach, which consists of defining the dynamic equilibrium of an isolated semi-cylinder defined above the boundary of each boundary element. Their derivation was presented by Tadeu et al. [17].

3.3 Combined dual BEM formulation

The two boundary elements formulations just described can be combined to address the same problems and the case of a thin elastic inclusion. This is achieved by loading part of the boundary surface with monopole loads, and the remaining part with dipole loads. The thin solid bodies will then be solved by means of a closed surface.

4 Verification of the BEM formulations

The BEM results are verified by comparison with the analytical results for a scattered wavefield generated by an elastic cylindrical inclusion [5]. The point harmonic load exciting the host medium is applied at \(O (0.0 \text{ m}, -0.125 \text{ m}, 0.0 \text{ m})\) and two lines of receivers are placed in the host medium, at \(R_1 (0.075 \text{ m}, -0.025 \text{ m})\), and inside the inclusion, at \(R_2 (0.025 \text{ m}, 0.025 \text{ m})\), as illustrated in Figure 1a. The responses given by the BEM models are successfully compared with the analytical results in Figures 1b-c, when \(k_c = 25 \text{ rad/m}\). Both the real and imaginary parts of the responses can be observed, with the analytical solution represented by solid and broken lines.
5 Numerical examples

The combined Dual BEM formulation is used to assess the scattering of elastic waves generated by two thin 2D inclusions filled with different solid materials, placed in an unbounded host medium. In the first case, a weaker 5 mm thick solid inclusion is made of cork (180 kg/m³), allowing the propagation of dilatational and shear waves with velocities of 288.7 m/s and 204.1 m/s, respectively. In the second example, the same thin inclusion is now made of steel (7850 kg/m³), which permits the corresponding wave propagation velocities of 5970.0 m/s and 3191.1 m/s, respectively. The host elastic medium is homogeneous (2140 kg/m³) and allows a dilatational wave velocity of 2696.5 m/s and a shear wave velocity of 1451.7 m/s. A harmonic line load is placed at point O and excites the elastic medium near the 2D inclusions (Figure 2).

The TBEM formulation is used to discretize the upper part of the inclusion's surface and the BEM formulation is applied to discretize the lower part. A relation between the wavelength (λ) and the length of the boundary elements (L), is set equal to 10 to determine a suitable number of boundary elements. Computations are performed in the frequency domain from 2000 Hz to 256000 Hz.
with a frequency increment of 2000 Hz. The time responses are modelled by a Ricker pulse with a characteristic frequency of 75000 Hz. The horizontal thin inclusions are 5 mm thick and their extremities are represented by 5 mm diameter semi-circumferences (see Figure 2 for global geometry and a close-up of the round extremities).

Figure 2: Elastic 5 mm thick horizontal inclusion: problem geometry.

The computations are performed in grids of evenly spaced receivers along \( x \) and \( y \) directions at 0.003 m. The displacement components in the \( y \) direction scattered by the softer and harder thin inclusions are shown at different time instants in Figure 3.

Figure 3: Elastic scattering by different 5 mm thick horizontal inclusions: \( y \)-component displacements \( (u_y) \) at two time instants.
The waves exciting the elastic medium propagate in all directions. When the incident pulses hit the elastic inclusions they are partly reflected back as P- and S-waves. Some energy is transmitted to the opposite side of the inclusions after passing through the cork and steel media as P- and S-waves. These waves generate multiple reflections on the inclusions’ upper and lower surfaces. In this process P- and S-waves are generated simultaneously, and these propagate away from the inclusion in the elastic medium.

The main differences in the observed wave fields are related to the dilatational and shear wave propagation velocities in the solid media inside both horizontal inclusions. For the case of the softer inclusion, a significant number of the incident pulses are diffracted around the heterogeneity, and at the later time instants waves can still be seen trapped inside the slower elastic medium. For the case of the harder inclusion, the pulses that pass through the heterogeneity are slightly attenuated and the pulses visible inside the harder medium are quickly transmitted to the exterior elastic medium. It is interesting to notice that the phase shift (in relation to the incident pulses) of the pulses reflected back from the buried inclusions only occurs for the harder inclusion example. This is due to the relative differences in elastic wave propagation speeds between the host elastic medium and the solid materials that completely fill the thin inclusions.

6 Conclusions

A combined Dual BEM algorithm has been implemented in order to model the scattering of 3D elastic waves generated by thin elastic inclusions buried in an unbounded 2D host medium. This model is formulated in the frequency domain and expressed in terms of waves with different wavenumbers in the $z$ direction, considering the 2-1/2D geometry of the problem. The numerical model was successfully verified by comparing its displacement field, scattered by a circular cylindrical elastic inclusion, with results obtained analytically and by a BEM and a TBEM model. Two numerical examples are presented, where the wave scattering in the vicinity of two different elastic inclusions is computed by the combined Dual BEM proposed model.

References


