Transformative models in reliability assessment of structures

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Abstract

In this paper, the relationship “load-loading effects” of engineering structures from the standpoint of applied calculation models is investigated. These simulations, both theoretical and experimental, replace the actual construction and are expected to express its existing characteristics under the given load. They are used for the specification of the response of the structures from the angle of the reliability conditions in the groups of limit states of load-carrying capacity and serviceability.

As an example of transformative simulations, the comprehensive methodology for determining the probability of fatigue failure for mixed-mode fatigue is presented. The loading is mixed-mode with randomness in the initial crack length, final crack length, initial crack angle, initial crack location, fatigue crack growth parameters, and the applied stress. The approach consists of calculating the reliability index that is applied to determine the first-order probability of collapse, by solving a constrained optimization problem.

The helpfulness of the probabilistic approach for structural analysis is indicated, namely by employing perturbation procedures.

Keywords: best guess, built-up beam, carrying resistance, crack path discretization, dynamic load factor, reference value, reliability index.

1 Introduction

Transformation models change the loading into the response of an individual construction to the application of load. These structural versions have to represent, if applicable, the real construction. A choice of the model and the study of relevant random variables incorporated must reflect fabrication and
installation tolerances, and also the influences of both environment and non-negligible imperfections. Transformation models fit in the reliability assessment method according to [1], in compliance with fig. 1.

Selection of a transformation model reflects the progress of structural mechanics and of the experimental approaches.

Structural analysis is the process of determining the response of the structure due to specific loadings. Data related to the determination and application of structural analysis models may be generally considered to be random variables. Examples of these random variables are geometrical and physical properties of the structure, imperfections, damping due to non-structural components, environmental effects, and soil-structure interaction. The accuracy of the selected structural analysis model depends on the evaluation and application of these variables and on their interaction. The structural analysis model chosen must be appropriate for the response and should represent the intended behaviour of a structure as closely as possible up to the limit state under consideration. We can launch simplifications to make the problem workable.

![Figure 1: Reliability assessment process.](image)

2 **Overview of transformative simulations**

To model the real construction and its characteristics when subject to loading and environmental influences, the version of structural analysis may be theoretical, empirical, or half-empirical. A half-empirical model is based partly on experimental results and partly on theoretical hypothesis. With respect to the expected response of the structure to the loading, and in conformity with the
details of the actual problem, the model may be static or dynamic, elastic or elasto-plastic.

The structural mechanics model can consider first order or second order analysis, and also time dependence.

The leading mathematical and physical models accessible to the design engineer are illustrated in fig. 2.

Figure 2: List of contents of the leading transformation version.

Physical versions serve as a guideline to examine experimentally the resistance and the serviceability. Tests may replace an analytical or numerical approach especially if adequate calculation models are not available, if a large number of similar components will be used, or if the real behaviour of a structure is of special interest. The following are examples of the types of tests that may be performed: tests to establish directly the ultimate resistance or serviceability properties of structural parts, tests to obtain specific material properties, tests to reduce uncertainties in transformation models (e.g. full size prototypes), control conformity tests to check the quality of mass-produced structures, and verification tests to check the behaviour of the actual structure after completion.

The issues may be applied to a definite structure or can function as a basis for the design of a broad range of construction, inclusive of the progress of rules in structural codes.
Mathematical models are of cardinal importance for constructional design usage. Such versions can be either analytical or numerical. Analytical models can be applied in everyday work such as in codes for structural design. In complex circumstances, numerical models can yield more suitable and detailed outcomes.

The principle of analytical modelling is to express a problem as a system of equation and to tackle it in closed form. Analytical models and their solution methods involve a very wide range of structural problems including plates, beams or columns elements, frames, and local structural details. The solutions of more complex problems using analytical models, however, are limited to those problems with special boundary conditions such as, types of supports, loading etc. Analytical methods are advantageous in that they provide a good physical understanding of the interaction of the factors and effects involved. They allow for simplifications to be introduced in relation to the significance of the problems (i.e., simplified formulas in codes for practical applications). In addition, they are simple enough for relatively inexpensive calculating tools like calculators. Analytical models can accelerate probabilistic simulation-based reliability assessment.

For problems that cannot be solved analytically, or in other adverse occurrences, one can use various numerical methods which are continually being improved. In so far as a given problem can be expressed by differential equations but cannot be solved in closed form, possibly because of the given boundary conditions, numerical solutions can be used. For these continuous models variational methods, finite difference methods, and other numerical procedures are available. These models were popular some time ago, when computers were already capable of resolving quite extensive systems of linear algebraic equations, but both the finite element method and boundary element technique had not yet been developed to their present-day perfection. Fair experience has been musteried in a similar way with other analogous algorithms, for example the collocation approach.

The present-day methods based on idealizing the construction by discrete elements are represented primarily by the FEM and BEM. Some other methods like finite strips and the folded plate method are available, however not extensively used nowadays. Today, the concept of the finite element method is frequently used. Major general aimed analysis programs have been worked out employing displacement-based finite element method.

2.1 Static model

(a) Time-independent loading.

In the case of time-independent loading, the equilibrium condition may be written down by a set of equations, in the form

$$K \times U = F$$

where the stiffness matrix $K$ defines the load carrying system, the vector $F$ defines the loading acting on the structure, and $U$ is the function representing the
resulting responses of the structure to the loading. Response can be expressed by deformations and corresponding stresses, moments, axial forces or variety of other quantities such as reactions at the supports. The solution to eqn. (1) means the static response of the structure to the loading.

(b) Time-dependent loading

As stated in the previous section, static models fit well with time-independent loading. Often, time-dependent loading causes dynamic response. If the forces are time-dependent, the designer must consider problems in which the inertia of accelerating masses must be taken into consideration. However, in some situations static response may be time-dependent. Such a situation can occur when the dynamic component of the response of a structure is negligible but loading is time-dependent. Basically, this consideration of inertia distinguishes the dynamic response from the static response. This is the case in many common situations in structural design where only “static” analysis is conducted and the response history need not be considered at all, or the response history is based on static analysis at selected points in time. Employing a point-in-time analysis, for special circumstances and if demanded, a static response description may be gained and applied such as in assessing fatigue damage accumulation and risk of fatigue collapse.

2.2 Dynamic models

In general, dynamic models applied to the analysis of dynamic response, may be rendered by means of the dynamic equilibrium equations

\[ M \times U'' + C \times U' + K \times U = F(t) \]  

where \( M \) stands for the mass matrix, \( C \) the damping matrix, \( K \) the stiffness matrix of the structural system, \( F(t) \) is the vector of the time-dependent loading function; \( U \) is function representing the resulting response of the structure to the loading (for example deformations), and \( U' \) and \( U'' \) are the first and second derivatives of the function \( U \). In this universal instance, the response of the construction is dynamic, and equation (2) is named the equation of motion.

In the majority of cases, dynamic response of a structure is created by a fast alteration of the magnitude, position or direction of the loads. However, a sudden change of stiffness or failure of a structural element such as the fracture of cables of a guyed mast or of a cable stayed bridge can gives rise to dynamic response as well.

Dynamic models may be formulated in the time domain or in the frequency one. When the load history is described in statistical terms, a statistical description of the response is also desirable. Based on such models, one can determine the probability of exceeding some limit state in a given reference period. Structural properties may be random as well as deterministic and time dependent as well as time independent. In a full probabilistic analysis, the variation of structural properties is taken into account. The models for dynamic
analysis consist of stiffness, inertia and damping models. Applying to the stated models, for dynamic response in severe earthquakes, a ductility control and corresponding hysteretic energy dissipation may be essential to clarify cyclic degradation.

2.3 Quasi-dynamic models

Frequently, dynamic models are replaced by quasi-dynamic versions. At the same time, the time dependent loading $F(t)$ (see eqn. 2) is replaced by an “adjusted” time-independent load $F$ resulting in approximately the same response $S$ (i.e., moments, stresses, deflections and others). We may formulate this approach by the following relationships, with respect to the response, $S$, of a construction to the loading:

$$
\frac{\text{dynamic model}}{\text{quasi-dynamic model}} = S_{\text{dyn}}(t) = H_{\text{dyn}} \times F(t)
$$

$$
S_{\text{q-dyn}} = H_{\text{stat}} \times F \times DF
$$

where $S_{\text{dyn}}(t)$ – the dynamic response, and $S_{\text{q-dyn}}$ – the quasi-dynamic response of the structure to the loading, $H$ are the transformation operators (static or dynamic) containing parameters of the load carrying system. $F(t)$ is the time-dependent loading, $F$ is the time-independent loading, and DF is the “Dynamic Load Factor”. The DF is defined (usually) as the ratio of the dynamic deflection at any time to the deflection which would have resulted from the static application of some selected arbitrarily value of the load which is used in specifying the load-time variation.

DF depends on the natural frequency and the relative damping. The dynamic factor approach may be used to determine the extreme values of the response, although the response history cannot be obtained. The application of deterministic values of DF as given in specifications (which reflect both, the loading history as well as the properties of the structure), when not regarded conveniently, can result in considerable variances from the real dynamic response to the loading.

3 Fatigue fracture in the light of reliability

Advances in fracture mechanics have facilitated to qualify a cardinal problem in lifetime prediction: the reliability, or probable life, of a flawed construction. By a flawed construction, we understand a structure weakened by a crack below the threshold of elucidation by inspection or a crack of a known size.

For life expectance determination of flawed construction, the influence of the geometry of the component or structure and its interaction with the propagating crack should be contemplated. Alternatively, the effect of the state of stress that is really present in the body and its interaction with the singular stress state at the crack-tip are not precisely considered in the reliability appraisal.

There are two main difficulties to calculate the reliability of flawed construction by truly simulating the propagating crack in the structure: (i) a lack
of data on the distributions of uncertain variables; and (ii) the great computational onus that is associated with the resultant probabilistic analysis.

The probabilistic finite element method (PFEM), in conformity with Besterfield et al. [3], provides a tool for estimating the effects of uncertainties in loading, material properties, and geometry on the uncertainties in the response variables. Via a fusion of the probabilistic finite element method with fracture mechanics, it has become possible to estimate the statistics of the stress intensity factor at the crack-tip.

Fatigue crack growth is sensitive to many characteristics and these parameters can rarely be stipulated accurately.

Uncertainties in the crack geometry, material properties, crack direction, crack growth, component geometry, and load time history, all play an important part. In this way, the prediction of fatigue collapse must be being an interpreted probabilistic problem. Several authors have recently studied probabilistic fatigue crack growth from a theoretical and statistical viewpoint using classical problems with known deterministic solutions.

Generally, for probabilistic fatigue crack growth, randomness in crack geometry, material properties, crack direction law, crack growth law, component geometry, and load time history are notable.

The reliability problem for fatigue crack growth is formulated by dint of an optimization procedure. In the optimization problem, the equilibrium equation and crack direction law at each discretized crack point are asserted by Lagrange multipliers. The solution methods employing PFEM and adjoint methods are then discussed. Finally, some detailed computational aspects are given.

Figure 3: Comparison of the fatigue life between the reference and FEM solutions.
3.1 Fatigue crack growth optimization

Above all, the crack direction law has to be discretized into “n” points along the crack path. At each discretization point, the crack direction equals, as follows

\[ z_k = \theta_k^T \kappa_k = 0 \quad k = 1, \ldots, n \]  \hspace{1cm} (4)

\( \kappa \) and \( \theta \) mean vectors of both the stress intensity factors and the angle between the tangent to the crack-tip and the x-axis, where \( \kappa_k \) and \( \theta_k \) stand for \( \kappa \) and \( \theta \) interpreted at \( \xi = z_k, k = 1, \ldots, n \). Like this, at each crack path discretization point, the new crack direction is recalculated and the crack is next permitted to propagate to the next point.

3.1.1 Optimization problem statement

The determination of the reliability index by the first-order probability theory is represented being a constrained optimization problem. The resultant nonlinear programming problem consists of determining the value of the primitive correlated random variables, \( b \), and the generalized displacements, \( \delta_i, i = 1, \ldots, n \), which minimize the distance from the origin to the limit-state surface in the independent standard normal space. The minimum distance from the origin to the limit-state surface in the independent standard normal space is called the reliability index, \( \beta \), i.e.

\[ \beta = r^T r \]  \hspace{1cm} (5)

where \( r \) is a vector of independent standard normal random variables.

The minimization is submitted to the equality constraints being based on the crack growth direction law and equilibrium, i.e., eqn. (4) and further

\[ K_i \delta_i = f_i \quad \text{no sum on } i, \quad i = 1, \ldots, n \]  \hspace{1cm} (6)

respectively, where \( K_i \) and \( f_i \) are the enriched stiffness matrix and external force vector, and the generalized displacements have the form

\[ \tilde{\delta}_i = \left[ \begin{array}{c} d_i \\ \kappa_i \end{array} \right] \quad i = 1, \ldots, n \]  \hspace{1cm} (7)

where \( d \) denotes the usual nodal displacement vector. An enriched element approach, which has the near crack-tip located singular strain field, is employed to obtain the eqns (6) and (7).

The minimization is also subject to the following inequality constraint

\[ T - T_s \leq 0 \]  \hspace{1cm} (8)
(i.e., the performance function being on the limit state surface is a constraint in the optimization problem).

4 Uncertainty simulation perturbation technique

Generally, uncertainties can emerge in material properties, specified surface values, the geometry of the body, or any combination of these factors. When extending the formulation to embrace uncertainties, we differentiate between various classes’ problems (see [4]). The philosophy of randomness or uncertainty may be distinguished like this:

(i) uncertainty in view of system parameters leads to a “spread” or density of possible parameter values, i.e., random variable simulation

(ii) uncertainty with regard to the behaviour of parameters as time-dependent processes, where the statistics of each process may be either-time-dependent or –independent, i.e., random or stochastic function model, and

(iii) uncertainty as to the characteristics of parameter properties as functions of spatial coordinates where, again, the statistics of the functional behaviour may be either space dependent or independent, i.e., random or stochastic field model.

An instance of considerable interest in engineering is one where the statistics of the material parameters $\mu$ and $\nu$ are available in the way that their mean values are known and deviations relative to the mean are not too large. We limit this problem to a body of one material with given surface values and geometry. Afterwards, the unknown surface displacements and tractions can be expanded in a Taylor series about the mean values.

4.1 Beam stiffness

The definition of two independent material properties, for the most part $\mu$ and $\nu$ (the shear modulus and Poisson’s ratio), is needed for linear elastic analysis of beams. There exist several circumstances when an accurate definition of $\mu$ and $\nu$ is impossible. This ends in great difficulties in the analysis routine. Such events are e.g.: (i) reinforced (or plain) concrete beams, (ii) equivalent (built up) beams.

<table>
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<th>$\bar{\mu}$</th>
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<th>$\mu_2$</th>
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5 Conclusion

For engineering problems both, theoretical and experimental, the transformation models serve as a tool to determine the response of the structure to loading. A selection of the model depends on the requirement of a solution, the accuracy required and on the nature of the reliability condition (from the standpoint of the carrying capacity or serviceability), and suchlike. The assignment of the scientific field – structural mechanics – is to work out and improve the
simulations in the way that they faithfully replicate the actual effectiveness of the systems, components and elements of the structures and efficiently allow to monitor the relation “loading-response” for the construction.

The technique presented to derive fatigue crack growth reliability produces very useful outcomes, is computationally effective, and may be extended to constituents of many forms. The method is suitable for mixed-mode fatigue crack growth. The aptitude of this method to stipulate the probability of fatigue collapse owing to uncertainties in the element geometry, applied loads, material properties, and crack geometry is of primary importance to the design engineers of various specialist fields. An inherent shortcoming of the methodology is that a very accurate calculation of the stress intensity factors is necessary so as to obtain reasonable precision for the reliability index. The results demonstrated prove the initial crack length to be a critical parameter. Taking into account that crack lengths below the threshold of a visitation limit are assumed to show a considerable quantity scatter, this makes it imperative that the life expectancy of a construction be interpreted from a stochastic standpoint. But still, except for some difficulties that may emerge when remeshing, the procedure is practicable to arbitrary geometries.

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References


