BE DRM-MD for two-phase flow through porous media

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Abstract

The numerical model of two-phase flow through porous media is used in many branches of science and engineering, including groundwater hydrology, soil science, reservoir problems in petroleum engineering, storage of radioactive or chemical wastes, etc. Two-phase flow consists of two flowing fluids, which do not interchange mass and do not have reaction with the solid matrix. The equations governing the two-phase flow through porous media are special cases of general balance laws. From a numerical point of view, due to the high nonlinearity of the governing equations and their strong coupling, the simulation provides a challenging numerical problem, even when simplifying assumptions are invoked. A new numerical scheme based on the dual reciprocity method – multi domain (DRM-MD) for the numerical simulation is presented. The efficiency of the method is proved on one and two dimensional two-phase flow examples.

1 Introduction

Numerical simulation of multiphase flow through fractured rocks is important in many branches of science and engineering. Hydrologists, agricultural engineers, soil physicists and others have for many years been concerned with fluid movement in air-water porous media systems. In the most recent years, environmental concerns and search for secure atomic waste depositories and for remediation strategies for contaminated aquifers raised more interest in the movement of fluids in the so-called vadose or unsaturated zone interposed between the atmosphere and the groundwater (saturated zone). Many potential
groundwater contaminants are introduced at (or near) the soil surface via atmospheric deposition, spills, leakage from underground tanks, subsurface waste disposal, etc. Furthermore, a wide class of environmental contaminants consists of organic compounds of low water solubility which can occur as a separate liquid phase in the soil. Such liquids include many widely used industrial solvents and automobile and jet fuels which unfortunately often enter the ground via surface spills or leaks from underground storage tanks. Historically, the greatest stimulus for development of multiphase flow models was initiated by the commercial interest of the petroleum industry, induced by the lure of more efficient oil and gas recovery from reservoirs. Most of the existing flow and transport models are single-phase models that describe the groundwater flow and the convective-dispersive spreading of one or more components entirely dissolved in water. The unsaturated zone is a multiphase system, consisted of wetting phase (water), air and non-wetting phase (for example, oil). Two-phase flow consisted of two flowing fluids, which do not interchange mass and do not have reaction with the solid matrix, is the simplest multiphase flow. The elimination of the equation of the gas phase is often referred to as Richard's approximation and is the basis of conventional analyses of two-phase flow. The porous media itself is assumed to be incompressible.

2 Governing equations

The equations governing the two-phase flow of fluids are special cases of general balance laws. When two fluids coexist in a porous medium, one of them will generally have preferential wettability for the solid phase and will occupy the smaller voids (wetting phase, subscript W), while the less wetting fluid is consigned to larger voids (non-wetting phase, for example oil, subscript O). Darcy’s law holds for both the phases, see [1-3]:

\[
\frac{\partial}{\partial t}(S_w \rho_w) - \nabla \left[ \rho_w \lambda_w K (\nabla p_w - \rho_w g) \right] - \rho_w q_w = 0
\]

\[
\frac{\partial}{\partial t}(S_o \rho_o) - \nabla \left[ \rho_o \lambda_o K (\nabla p_w + \nabla p_{cow} - \rho_o g) \right] - \rho_o q_o = 0
\]

where:

- \( S_\alpha \) - saturation of the fluid phase \( \alpha \);
- \( \rho_\alpha \) - density of the fluid phase \( \alpha \);
- \( p_\alpha \) - pressure head in the fluid phase \( \alpha \);
- \( \lambda_\alpha = k_{ra} / \mu_\alpha \) - mobility of the fluid phase \( \alpha \);
- \( k_{ra} \) - relative permeability of the fluid phase \( \alpha \);
- \( \mu_\alpha \) - dynamic viscosity;
- \( K \) – permeability.

Equations (1) and (2) are coupled, since they have to satisfy the condition that the fluids fill up the pore volume, i.e. the additional relations:

\[
S_w + S_o = 1
\]

A pressure difference occurs along the fluid-fluid interface, whose magnitude depends on the interface curvature. This pressure difference is termed as capillary
pressure $p_{\text{cow}}$ and it must be positive by definition, since the non-wetting phase pressure must exceed that in the wetting phase:

$$p_{\text{cow}} = p_o - p_w$$  \hfill (4)

Darcy’s law for the wetting and non-wetting phase can be formulated as follows:

$$v_w = -\lambda_w K (\nabla p_w - \rho_w g)$$  \hfill (5)

$$v_o = -\lambda_o K (\nabla p_w + \nabla p_{\text{cow}} - \rho_o g)$$  \hfill (6)

Introducing (5) and (6) into equations (1) and (2) one can obtain shortened form:

$$\phi \frac{\partial (S_w \rho_w)}{\partial t} + \nabla [\rho_w v_w] - \rho_w q_w = 0$$  \hfill (7)

$$\phi \frac{\partial (S_o \rho_o)}{\partial t} + \nabla (\rho_o v_o) - \rho_o q_o = 0$$  \hfill (8)

Summing up the equations (7) and (8), and introducing (3):

$$v_i = v_w + v_o$$

$$\nabla v_i = \nabla (v_w + v_o) = q_w + q_o - \frac{1}{\rho_o} v_o \cdot \nabla \rho_o - \frac{1}{\rho_w} v_w \cdot \nabla \rho_w$$

$$+ \phi \left[ \frac{1-S_w}{\rho_o} \frac{\partial \rho_o}{\partial t} + S_w \frac{\partial \rho_w}{\partial t} \right] = 0$$  \hfill (9)

Neglecting the compressibility of both phases yields to:

$$\nabla v_i = q_w + q_o = q_i$$  \hfill (10)

The combination of equations (5) and (6) when the first one is multiplied by $\lambda_o$, and the second one by $\lambda_w$, leads to:

$$v_w \lambda_o - v_o \lambda_w = \lambda_o \lambda_w K (\nabla p_{\text{cow}} - \rho_o g + \rho_w g)$$  \hfill (11)

with

$$v_o = v_i - v_w$$

$$v_o = \frac{\lambda_o}{\lambda_o + \lambda_w} [v_i - \lambda_o K (\nabla p_{\text{cow}} - \rho_o g + \rho_w g)]$$  \hfill (12)

After substituting eq. (12) in eq. (8), the resulting equation is:

$$\nabla \left[ f_o v_i - \lambda (\nabla p_c + \rho_w g - \rho_o g) \right] = \phi \frac{\partial S_w}{\partial t} - q_o$$  \hfill (13)

where:

$$f_o = \frac{\lambda_o}{\lambda_o + \lambda_w}$$ - fractional flow for the oil \hfill (14)

$$f_w = \frac{\lambda_w}{\lambda_o + \lambda_w}$$ - fractional flow for the water \hfill (15)

and

$$\lambda = \frac{\lambda_w \lambda_o}{\lambda_o + \lambda_w}$$  \hfill (16)
All the unknowns in the equation (13) are functions of the saturation, thus one can use following expressions for the derivatives:

\[
\begin{align*}
\frac{\partial p_c}{\partial x_i} &= \frac{dp_c}{ds_w} \frac{\partial s_w}{\partial x_i} \\
\frac{\partial (f_0 v_i)}{\partial x_i} &= v_i \frac{\partial f_0}{\partial s_w} \frac{\partial s_w}{\partial x_i} - f_0 q_i \\
\frac{\partial}{\partial x_i} \left[\lambda (\rho_w g - \rho_o g) \right] &= (\rho_w g - \rho_o g) \frac{d \lambda}{ds_w} \frac{\partial s_w}{\partial x_i}
\end{align*}
\]  
(17)

Including (17), (18) and (19) into equation (13) yields the following equation:

\[
\begin{align*}
\frac{\partial}{\partial x_i} \left[\lambda \frac{dp_c}{ds_w} \frac{\partial s_w}{\partial x_i} \right] - \left[ v_i \frac{df_w}{ds_w} + (\rho_w g - \rho_o g) \frac{d \lambda}{ds_w} \right] \frac{\partial s_w}{\partial x_i} \\
- \phi \frac{\partial s_w}{\partial t} - q_w + f_w q_i &= 0
\end{align*}
\]  
(20)

The first term in the two-phase saturation equation (20) represents dispersive part, the second term is convective part, the third is mass storage term and last ones a source and sink terms. The equation is parabolic because of the presence of the term that includes capillary pressure.

Neglecting only the gravitational effects and the source and sink term, for simplicity, and taking the significance of the capillary pressure gradient into account, one arrives to the so-called Mc Whorter problem:

\[
\phi \cdot \frac{\partial s_w}{\partial t} + q_i \cdot \frac{df_w}{ds_w} \cdot \frac{\partial s_w}{\partial x_i} = - \frac{\partial}{\partial x_i} \left( D \cdot \frac{\partial s_w}{\partial x_i} \right)
\]  
(21)

\[
D = k_w \cdot K \cdot \frac{f_w}{\mu_o} \cdot \frac{dp_c}{ds_w} - \text{dispersion tensor}
\]  
(22)

Assuming that all the properties dependent on water saturation (the dispersion tensor) are constant in a particular time step, equation (21) can be written as:

\[
D \cdot \nabla^2 s_w = -\phi \cdot \frac{\partial s_w}{\partial t} - q_i \cdot \frac{df_w}{ds_w} \cdot \frac{\partial s_w}{\partial x_i}
\]  
(23)

A critical component of predictive two-phase flow models is the mathematical description of the relative permeability, saturation and pressures relations (k-S-p), sometimes referred to as constitutive models. Reliable experimental measurements are difficult to obtain, they are time consuming, expensive and require advanced laboratory skills. A summary of different constitutive models is given by Helmig, see [3].

3 Numerical implementation

The numerical technique that is used to solve the partial differential equation defined by the model is the Boundary Element Dual Reciprocity Method – Multi
Domain scheme (BE DRM-MD) [4–8]. It has been used for the first time to solve a two-phase flow model. The method belongs to the boundary element techniques that transform the original partial differential equation into an equivalent integral equation by means of the corresponding Green’s theorem and its fundamental solution. Simplicity, elegance of the formulation and its efficient solution extended the application of the BEM to a wide variety of time dependent and non-linear problems.

The governing equation that describes a linear time-dependent flow in porous media, in the general form can be written as,

\[ \nabla^2 u = b(x, y, u, t) \text{ in the domain } D \]  \hspace{1cm} (24)

with the ‘essential’ boundary conditions of the type \( u = \bar{u} \) on \( \Gamma_1 \) and ‘natural’ boundary conditions \( q = \partial u / \partial n = \bar{q} \) on \( \Gamma_2 \). Here \( \Gamma = \Gamma_1 + \Gamma_2 \) is the exterior boundary that encloses the domain \( D \) and \( n \) is its outward normal.

For the two-phase flow model, \( u \) represents \( S_w(x, t) \) in the equation (23) for the flow and \( b = -\phi \cdot \frac{\partial S_w}{\partial t} - \frac{q_i}{D} \cdot \frac{df_w}{dS_w} \cdot \frac{\partial S_w}{\partial x_i} \).

Applying the DRM (Dual Reciprocity Method) approach to the equations (23), according the detailed explanation in references [6] and [7], yields:

\[ HS_w - GQ = - \frac{S}{D} \left[ \phi \cdot \frac{\partial S_w}{\partial t} + q_i \cdot \frac{df_w}{dS_w} \cdot \frac{\partial S_w}{\partial x_i} \right] \]  \hspace{1cm} (25)

where:

\[ S = (HS_w - GQ) \cdot F^{-1} \]  \hspace{1cm} (26)

The two boundary element characteristic matrices \( H \) and \( G \) on both sides of the equation (25) consist of coefficients, which are calculated assuming the fundamental solution is applied at each node successively, and depending only on geometrical data, see [4–7].

Applying BE DRM derivation:

\[ \frac{\partial S_w}{\partial x_i} = \frac{\partial F}{\partial x_i} \cdot F^{-1} \cdot S_w \]  \hspace{1cm} (27)

Including that:

\[ T = S \cdot \frac{1}{D} \cdot q_i \cdot \frac{df_w}{dS_w} \cdot \frac{\partial F}{\partial x_i} \cdot F^{-1} \]  \hspace{1cm} (28)

and

\[ koeff = \frac{\phi}{D} \]  \hspace{1cm} (29)

the main equation takes the following form:

\[ HS_w - GQ = -S \cdot koeff \cdot \frac{\partial S_w}{\partial t} - T \cdot S_w \]  \hspace{1cm} (30)

\[ (H + T)S_w - GQ = -S \cdot koeff \cdot \frac{\partial S_w}{\partial t} \]  \hspace{1cm} (31)
For the time marching scheme:

\[
S_w^n = (1 - \theta_s) \cdot S_w^n + \theta_s \cdot S_w^{n+1}
\]  
\[
q^n = (1 - \theta_q) \cdot q^n + \theta_q \cdot q^{n+1}
\]

where \( \theta_s \) and \( \theta_q \) take values between 0 and 1. Time derivative is approximated using a finite-difference approximation:

\[
\frac{\partial S_w}{\partial t} = \frac{1}{\Delta t} \cdot (S_w^{n+1} - S_w^n)
\]

Finally the equation for two-phase Mc Whorton flow is:

\[
\left[ \theta_s (H + T) + \frac{k_{\text{coef}} \cdot S}{\Delta t} \right] \cdot S_w^{n+1} - G \cdot \theta_q \cdot Q^{n+1} =
\]

\[
= \left[ \frac{k_{\text{coef}} \cdot S}{\Delta t} - (1 - \theta_s) (H + T) \right] \cdot S_w^n + G \cdot (1 - \theta_q) \cdot Q^n
\]

At each interface between the adjacent sub-domains, the corresponding full matching conditions are imposed. At the \( m \)th interface of two adjacent sub-domains, \( i \) and \( i+1 \), the saturation needs to be continuous and the flux leaving one sub-domain has to be equal to the flux entering the other sub-domain:

\[
S_{wi}^i = S_{wi+1}^{i+1}
\]

for saturation

\[
\left( \frac{\partial S_{wi}}{\partial n} \right)_m = -\left( \frac{\partial S_{wi+1}}{\partial n} \right)_m
\]

for saturation flux

where subscripts \( i \) and \( i+1 \) indicate the sub-domain, \( n_i \) and \( n_{i+1} \) represent the unit outward vectors normal to the interface looking from the sub-domain \( i \) and \( i+1 \) respectively.

DRM-MD scheme has been implemented in a code which offers high flexibility in mesh generation. The mesh can be built of sub-domains of various sizes, geometries and numbers of boundary elements per sub-domain. Coupling of large homogeneous domains that do not need to be discretized with very small domains is implemented in the code, possibility that had been previously obtained only with BEM-FEM formulations. The present formulation removes problems with the matching compatibility of the two methods.

The sub-domain formulation leads to block banded matrix systems with one block for each sub-domain and overlaps between blocks when sub-domains have a common interface. In the limit of a very large number of sub-domains the resulting internal mesh pattern looks like a finite element grid. One obvious advantage of the DRM-MD is that the matrix multiplications are done on the level of each sub-domain, accelerating this way the numerical scheme.

By using the DRM-MD one is able to solve the system for the field variables, in this case \( S_w \), and their derivatives. The DRM-MD approach requires no major restrictions in the size and geometry of sub-domains.
4 Model verification

Two examples are given for model verification in this paper. For both of them, the gravitational effects are neglected. One of the most popular models, proposed by Brooks and Corey is employed for description of the relative permeabilities and capillary pressure constitutive relationships. Expression for the wetting phase, the power-law function, is:

\[ K_{rw} = \left( S_w^w \right)^{2+\lambda \frac{3}{2}} \]  

(37)

Similarly for the non-wetting phase:

\[ K_{row} = \left( 1 - S_w^w \right)^2 \left( 1 - S_w^{2+\lambda} \right) \]  

(38)

\[ p_{cov} = p_d \cdot S_w^{1/\lambda} \]  

(39)

where \( p_d \) is the bubbling pressure, which apparently represents the smallest capillary pressure at which a continuous non-wetting phase starts to enter the system.

The Brooks-Corey function is constrained by \( P_e \geq p_d \), see [9], neglecting capillary pressure variation with saturation below the entry pressure. For the most often used values of \( \lambda=2 \), the above equations obtain the following form:

\[ k_{rw} = S_w^4; \quad k_{ro} = \left( 1 - S_w^w \right)^2 \left( 1 - S_w^2 \right); \quad p_{cov} = p_d \left/ \sqrt{S_w} \right. \]  

(40)

![Figure 1: Brooks – Corey model; a) relative permeabilities; b) capillary pressure.](image)

According the definition, the water saturation cannot be greater than 1 or negative. In this work both phases are assumed to be mobile and their ranges of saturations [0,1] cover the two-phase flow region.
Because there are not available analytical solutions for complex nonlinear processes, the results of these test cases can only be checked for plausibility and compared with other numerical results in the literature.

4.1 One-dimensional two-phase flow example

In a one-dimensional set-up the initial oil filling is extracted on one side and water is entering the system on the other side. The numerical example that corresponds to the example in the book of R. Helmig, see [3] is analyzed. Domain is long \( L = 2.6 \text{m} \). Dirichlet boundary conditions are imposed: at \( x = 0.0 \text{m} \), \( S_w = 1.0 \) and at \( x = 2.6 \text{m} \), \( S_w = 0.01 \). These boundary conditions are chosen for sake of having a nonconductive right boundary. Initial saturation through the whole domain is \( S_w = 0.01 \). Residual saturations for the water and the oil are \( S_{wr} = S_{or} = 0.0 \).

Densities are \( \rho_w = \rho_o = 1000 \text{ kg/m}^3 \) and dynamic viscosities are \( \mu_w = \mu_o = 0.001 \text{ kg/(ms)} \) for both fluids. Properties of the rock are: absolute permeability \( K = 10^{-10} \text{ m}^2 \), porosity \( \phi = 0.3 \). The influence of the space discretization is analyzed with three meshes; the first one with 13 sub-domains of length 0.2m, the second mesh with 26 sub-domains of length 0.1m, and the third one is composed of 52 sub-domains x 0.05m. The results of the simulation with the grid with 52 sub-domains and time step \( dt = 0.5 \text{sec} \) are presented in Figure 2.

![Saturation profiles at four time steps (52 sub-domains, dt=0.5sec).](image)

Even the coarse grid simulation produces useable and accurate solution, though the results on the finer grids are considerably better. It is oscillation free and shows a good mass balance. Testing simulation with different time steps on each of these meshes showed that one should be extremely careful with the choice of the time step when rather coarse mesh is used for the domain.

4.2 Two-dimensional two-phase flow example

The so-called “five spot example” discretized with square mesh 300m x 300m is analysed. The initial saturation of oil is assumed to be \( S_o = 1.0 \) and initial pressure
is $P_0=2\cdot 10^5$ Pa. Initial saturation through the whole domain is $S_w=S_o=0.0$. Residual saturations for the water and the oil are assumed to be $S_{wr}=S_{or}=0.0$. Densities are $\rho_w=\rho_o=1000 \text{ kg/m}^3$ and dynamic viscosities are $\mu_w=\mu_o=0.001 \text{ kg/(ms)}$ for both fluids. The properties of the rock are: absolute permeability $K=10^{-7} \text{ m}^2$, porosity $\phi=0.2$. Two cases are considered: one homogeneous with permeability $K_1=10^{-7} \text{ m}^2$ all over the domain and the second one is a problem with low permeability zone (in the center of the domain a zone of $112.5\text{m}\times 112.5\text{m}$ with low permeability of $K_2=10^{-10} \text{ m}^2$, three orders of magnitude less than in zone 1).

![Homogeneous and heterogeneous five spot example.](image)

**Figure 3:** Homogeneous and heterogeneous five spot example.

### 5 Conclusions

In this work two-phase flow model was presented for the flow through fractured porous rocks. Even the coarse grid simulation produces useable and accurate solution, though the results on the finer grids are considerably better. It is oscillation free and shows a good mass balance. Testing simulation with different time steps on each of these meshes showed that one should be extremely careful with the choice of the time step when rather coarse mesh is used for the domain. Both examples showed good agreement for the representation of the saturations. They perform very well for non-homogeneous domains, even when there are great differences between the permeabilities within the domain. The models were implemented using the DRM-MD scheme, which showed flexibility in the modelling that previously has been obtained using the FEM-BEM coupling.

It is important to point out that to assemble a coherent set of boundary and initial conditions, appropriate material relationships and to use the two-phase model appropriately, in-depth understanding of the physical processes is required, because of the strong nonlinear coupling.
References


