Integral equations for elastic problems posed in principal directions: application for adjacent domains

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Abstract

This article addresses a new type of boundary condition in plane elastic boundary value problems. Principal directions are given on a contour separating interior and exterior domains; the stress vector is continuous across the contour. Solvability of this problem is investigated and the number of linearly independent solutions is determined. Some special cases in which the problem is underspecified have been reported.

Keywords: plane elasticity, boundary value problems, principal directions, complex potentials.

1 Introduction

Classical boundary value problems, BVP, of the plane elasticity require one of the following surface conditions to be known on the entire boundary of a domain (see Muskhelishvili [1]): (i) stress vector; (ii) displacement vector; or (iii) certain combinations of stress and displacement components (mixed problems). In these cases the BVP is well posed and possesses a unique solution.

Galybin and Mukhamediev [2] and Galybin [3] considered different types of BVPs in which magnitudes of stresses, displacements or forces are not specified on the boundary. It has been shown that the BVPs of this type may have a finite number of solutions or be unsolvable. Solvability depends on the, so-called, index of singular integral equations (see, e.g., Gakhov [4]). It can be determined in every particular case from the analysis of principal directions (of the stress tensor), orientation of displacement or forces on the entire boundary of a considered domain.
This article investigates solvability of a new plane elastic BVP with a certain combination of boundary conditions, which has not been addressed before (sections 2 and 3). Namely, principal directions are given on a contour separating interior and exterior domains; the stress vector is continuous across the contour. The problem has direct applications in geodynamics for identification of stresses in adjacent tectonic plates. The article also presents some special (degenerated) cases in which the problem is underspecified (section 4).

2 Singular integral equation of the problem

2.1 Problem formulation in terms of stress functions

Let $\Gamma$ be a closed contour separating the complex plane into interior $\Omega^+$ and exterior $\Omega^-$ domains. Stress states in both domains can be expressed through sectionally holomorphic functions (complex potentials) $\Phi(z)$ and $\Psi(z)$ of complex variable $z=x+iy$ by the Kolosov–Muskhelishvili solution (no body forces)

$$ P(z,\bar{z}) = \Phi(z) + \bar{\Phi}(\bar{z}), \quad D(z,\bar{z}) = \bar{z}\Phi'(z) + \Psi(z) $$ (1)

Here the harmonic function $P$ (mean stress) and complex-valued function $D$ (stress deviator) represent the following combinations of stress components $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{xy}$.

$$ P(z,\bar{z}) = \frac{\sigma_{xx}(z,\bar{z}) + \sigma_{yy}(z,\bar{z})}{2}, \quad D(z,\bar{z}) = \frac{\sigma_{yy}(z,\bar{z}) - \sigma_{xx}(z,\bar{z})}{2} + i\sigma_{xy}(z,\bar{z}) $$ (2)

The stress vector on $\Gamma$ has the following complex form

$$ N^\pm(\zeta) + iT^\pm(\zeta) = P^\pm(\zeta) + \frac{d\zeta}{d\zeta} D^\pm(\zeta), \quad \zeta \in \Gamma $$ (3)

Hereafter a function of a single variable stands for the boundary value of this function, “$\pm$” denote the boundary values obtained by approaching $\Gamma$ from domains $\Omega^\pm$ respectively; $N^\pm$ and $T^\pm$ are normal and shear components of the stress tensor on $\Gamma$.

Principal directions of the stress tensor, angles $\varphi(z,\bar{z})$, are connected to the argument of the stress deviator, $\alpha(z,\bar{z})$, as follows

$$ \varphi(z,\bar{z}) = -\frac{1}{2} \alpha(z,\bar{z}), \quad \alpha(z,\bar{z}) = \arg D(z,\bar{z}) $$ (4)

The following boundary value problem, BVP, of holomorphic functions is considered further on: find stress potentials $\Phi(z)$ and $\Psi(z)$ by the following boundary conditions

$$ N^+(\zeta) + iT^+(\zeta) = N^-(\zeta) + iT^-(\zeta), \quad \zeta \in \Gamma $$

$$ \arg D^+(\zeta) = \alpha^+(\zeta), \quad \zeta \in \Gamma $$
where two real valued functions \( \alpha^\pm(\zeta) \) are known provided that boundary values of principal directions are given on the contour.

Boundary conditions (6) can also be expressed as follows
\[
\text{Im}\left[e^{-i\alpha^\pm(\zeta)}D^\pm(\zeta)\right] = 0, \quad \zeta \in \Gamma
\]  
(7)
As soon as potentials are found, the stress fields (i.e., stress functions and stress components) in both the exterior and interior domains can be determined by formulas (1) and (2).

2.2 Reduction to singular integral equations

Complex potentials satisfying (5) can be obtained from the representation provided by Savruk [5] in the following form
\[
\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g'(t)}{t-z} \, dt, \quad \Psi(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{g'(t)}}{t-z} \, dt - \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{t}g'(t)}{(t-z)^2} \, dt
\]  
(8)
Here function \( g'(t) \) is the derivative of a complex-valued function \( g(t) \) proportional to the jump of the displacement vector across the contour with the coefficient \( 2G(1+\kappa)^{-1} \) (\( G \) is the shear modulus, \( \kappa=3-4\nu \) for plane strain and \( \kappa=(3-\nu)/(1+\nu) \) for plain stress, \( \nu \) is Poisson’s ratio). It should be noted that the considered BVP does not require specification of elastic constants if one has no intention of analysing displacements.

Single valuedness of displacements imposes the following condition on \( g(t) \)
\[
\int_{\Gamma} g'(t) \, dt = 0
\]  
(9)
After simple transformations expressions of the stress functions take the form
\[
P(z, \bar{z}) = \text{Re} \left\{ \frac{1}{\pi i} \int_{\Gamma} \frac{g'(t)}{t-z} \, dt \right\}
\]  
\[
D(z, \bar{z}) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{g}'(t)}{t-z} \, dt - \frac{1}{2\pi i} \int_{\Gamma} \frac{(\bar{t}-\bar{z})g'(t)}{(t-z)^2} \, dt
\]  
(10)
These representations are valid for any point of the interior as well as exterior domains of the entire complex plane separated by for the contour \( \Gamma \).

Boundary values of holomorphic functions are found by the Sokhotski–Plemelj formulae \( 2\varphi^\pm = \pm g' + \mathbf{I}(g') \) (e.g., Gakhov [4]). Then for boundary values of the stress functions one obtains (see also [5] for the boundary values of the second integral in the expression for \( D \))
\[
P^\pm(\zeta) = \Re \left( \pm g'(\zeta) + \frac{1}{\pi i} \int_{\Gamma} \frac{g''(t) dt}{t-\zeta} \right) \tag{11}
\]
\[
D^\pm(\zeta) = \mp \frac{d\bar{\zeta}}{d\zeta} \Re(g'(\zeta)) - \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{g'(t) \bar{g}'(t)}{t-\zeta} + \frac{\bar{\zeta} - \zeta}{(t-\zeta)^2} \right) dt \tag{12}
\]

Substitution of the second formula in (11) into boundary conditions (7) results in the following system of singular integral equations

\[
\begin{align*}
\sin(2\theta + \alpha^+) \Re g' - \Im(e^{-i\alpha^+} J(g', \bar{g}')) &= 0 \\
\sin(2\theta + \alpha^-) \Re g' + \Im(e^{-i\alpha^-} J(g', \bar{g}')) &= 0
\end{align*}
\tag{13}
\]

where operator \(J(\ldots)\) denotes the integral term

\[
J(g', \bar{g}') = \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{g'(t) e^{-2i\theta(t)}}{t-\zeta} + \frac{\bar{\zeta} - \zeta}{(t-\zeta)^2} \right) dt
\tag{14}
\]

One can form a linear combination of both equations in (12) by excluding the complex conjugated operator from the system. This results in the following single complex equation

\[
\left( e^{i\alpha^-} \sin(\alpha^+ + 2\theta) + e^{i\alpha^+} \sin(\alpha^- + 2\theta) \right) \Re g' + \sin(\alpha^+ - \alpha^-) \ J(g', \bar{g}') = 0
\tag{15}
\]

Equation (14) is equivalent to the system of equation (12) if \(\alpha^+ \neq \alpha^- \neq 0, \pm \pi\) (this condition provides complex valuedness of (14)). In some cases of simple geometry it is not satisfied (see section 4).

Singular integral equation (14) is homogeneous therefore, according to Noether’s theorems [4], the number of its independent solutions is determined by the index of the problem (next section).

## 3 Solvability of integral equations

### 3.1 Reduction to the Riemann problem for holomorphic functions

Let us extract the dominant part of SIE (13) by separating singular and regular terms in the integral operator in (12), which leads to

\[
J(g', \bar{g}') = e^{-2i\theta} \ I(\Re g') + R(g', \bar{g}')
\tag{16}
\]

Here \(I(\ldots)\) and \(R(\ldots)\) are singular and regular operator respectively. They are expressed as follows (\(\zeta \in \Gamma\))
\[ I(g') = \frac{1}{\pi i} \int_{\Gamma} \frac{g'(t)}{t - \zeta} \, dt, \quad I^2(g') = g' \]  
(16)

\[ R(g', \bar{g}') = \frac{1}{2\pi i} \int_{\Gamma} \left( 2e^{-2i\theta(t)} - e^{-2i\theta(\zeta)} \right) \text{Re} g'(t) + \left( e^{-2i\arg(t-\zeta)} - e^{-2i\theta(t)} \right) g'(t) \, dt \]  
(17)

In the latter formula one can notice that the argument is bounded and continuous on an arbitrary smooth contour. It also satisfies the Hölder condition, and its value at the origin coincides with the angle of the tangent inclination to the \(x\)-axis, thus

\[ \lim_{t \to \zeta} e^{-2i\arg(t-\zeta)} = \lim_{t \to \zeta} \frac{\bar{t} - \bar{\zeta}}{t - \zeta} = \frac{d\zeta}{d\bar{\zeta}} = e^{-2i\theta(\zeta)} \]  
(18)

By substituting (15) into (14) and neglecting the regular integrals one obtains the dominant equation in the form

\[ \left( e^{i\alpha} \sin(\alpha^+ + 2\theta) + e^{i\alpha} \sin(\alpha^- + 2\theta) \right) \text{Re} g' \]

\[ + e^{-2i\theta} \sin(\alpha^- - \alpha^-) \quad I\left( \text{Re} g' \right) = 0 \]  
(19)

Solvability of (19) is determined by the coefficient of the correspondent Riemann problem that is found as follows (Gakhov [4])

\[ G(\zeta) = \frac{e^{i\alpha}(\zeta) \sin(\alpha^- (\zeta) + 2\theta(\zeta))}{e^{i\alpha}(\zeta) \sin(\alpha^+ (\zeta) + 2\theta(\zeta))} \]  
(20)

The index of a function is determined as its increment after the complete traverse of the contour in positive direction (counter clockwise) divided by \(2\pi\). It is evident that the index of \(G\) depends only on the difference of principal directions (because the ratio of the sines does not contribute into the increment of \(G\)), therefore

\[ \text{Ind} G = \frac{\alpha^+ (\zeta) - \alpha^- (\zeta)}{2\pi} \bigg|_{\Gamma} = -\frac{1}{\pi} \left( \varphi^+(\zeta) - \varphi^-(\zeta) \right) \bigg|_{\Gamma} = 2K \]  
(21)

where \(|_{\Gamma}\) denotes the increment.

Thus, for an arbitrary, simply connected domain, bounded by a smooth closed contour and for any non-negative index, \(2K\), the solution of the dominant equation will in general include a polynomial of \(2K\) order (or \(2K-1\) order if stresses vanish at infinity). This means that up to \(2K+1\) complex constants or (if each complex constant is counted as 2 real constants) \(4K+2\) real constants are included into the solution. For any negative index no bounded solutions exist.

This analysis has to be acknowledged in numerical implementation. Thus, after discretisation of (19) followed, for instance, by the collocation technique, the system for the determination of unknowns should have less rank then the number of unknowns (provided that \(2K\geq0\), which means that \(4K+2\) real parameters cannot be determined.
4 Degenerated cases

In some cases the operator $J(\ldots)$ is radically simple and the analysis of solvability should be revised. Such cases include simple geometries and/or special cases of load. Several special cases are considered in this section. All of them are special ones because normal stresses on the interface do not violate boundary conditions (7) and the potential $\Phi(z)$ can be determined with certain arbitrariness. Thus, the complete problem of stress tensor determination is underspecified and has an infinite number of solutions.

4.1 Joined half-planes

Let $\Gamma$ be the real axis, then $\zeta = x$ and $e^{i\theta} = 1$, which immediately results in

$$R(g',\bar{g}') = 0, \quad J(g',\bar{g}') = I(Re g'), \quad I(Re g') = -I(Re g')$$

Therefore the system of SIE takes the form

$$\begin{cases}
\sin \alpha^+ \Re g' - i \cos \alpha^+ I(Re g') = 0 \\
\sin \alpha^- \Re g' + i \cos \alpha^- I(Re g') = 0
\end{cases}$$

Both these two equations are of the dominant form. Their solvability depends upon indexes, $2\upk^+$ and $2\upk^-$, of the correspondent Riemann problems; these are determined as follows

$$2\upk^+ = \text{Ind} \left( e^{-2i\alpha^+} \right) = \frac{\Phi^+(x)}{\pi} \bigg|_{-\infty}^{\infty}, \quad 2\upk^- = \text{Ind} \left( e^{-2i\alpha^-} \right) = -\frac{\Phi^-(x)}{\pi} \bigg|_{-\infty}^{\infty}$$

The unknown function $\Re g'$ should satisfy both equations simultaneously, which is only possible if

$$\sin(\alpha^+ + \alpha^-) = 0$$

If boundary values of principal directions do not satisfy (25) only the trivial solution of the system exists, $\Re g' = 0$.

If (25) is satisfied, it follows from (24) that for solvability $2\upk^+ = 2\upk^- = 2\upk$ and solution of (23), in accordance with Gakhov [4], takes the form

$$\Re g'(x) = \cos \alpha(x) e^{\Lambda(x)} x^{-\upk} P_{2\upk-1}(x)$$

$$\Lambda(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \left( t^{-2\upk} e^{-2i\alpha(t)} \right)}{t-x} dt$$

where $P_{2\upk-1}(z)$ is a polynomial of order $2\upk-1$ with arbitrary complex coefficients (for the case when stresses vanish at infinity). Taking into account the fact that the imaginary part of $\Lambda(x)$ is constant due to the following integral

$$\int_{-\infty}^{\infty} \frac{\ln|t|}{t-x} dt = 2x \int_{0}^{\infty} \frac{\ln(t)}{t^2-x^2} dt = \frac{\pi^2}{2}$$
(Prudnikov et al., [6]) and one can notice that the coefficients of the polynomial may be considered as real provided that the imaginary part of $\Lambda(x)$ in (26) is omitted.

It is evident that neither trivial solution nor solution (29) allows one to identify $\Phi(x)$, because $\text{Im} \, \gamma$ (i.e., the density of crack opening displacements) can be chosen arbitrary.

### 4.2 Crack in the plane

Let a crack of length $2l$ be situated on the interval $(-l, l)$ in the complex plane. Stresses at infinity are assumed to be zero.

It is evident that the system derived for the case of half-planes remains. The analysis is also similar to that described above, however some corrections in the determination of the indices have to be introduced to account for the open contour.

The problem is reduced to the case of half-planes by putting $G(x)=1$ on $|x|>l$. Then the ends of the interval represent the points of discontinuity of $G(x)$. This becomes evident if the asymptotic behaviour of $D$ at the crack tips is considered.

Independent of the load it can be written in the form

$$
\sqrt{2\pi r} D = \left( K_i^\pm + 3iK_{II}^\pm \right) e^{-i\vartheta/2} - \left( K_i^\pm - iK_{II}^\pm \right) e^{-5i\vartheta/2}
$$

where $K_i$ and $K_{II}$ are stress intensity factors, the indices "$\pm$" refer to the right and left crack tips correspondingly and angle $\vartheta$ is the polar angle in local coordinate system $(r, \vartheta)$ with the origin at the crack tip. Now the argument of $\alpha=\alpha(x)$ can be calculated. In particular, for points lying near the tips of the crack, the argument of $D$ does not depend on $K_i$ and can be determined as follows

$$
\alpha(\pm l \mp 0) = \arg K_{II}^\pm = \frac{\pi}{2} \left( 1 - \text{sgn} K_{II}^\pm \right) \Rightarrow e^{2i\alpha(\pm l \mp 0)} = 1
$$

It is also seen that the argument $\alpha$ would gain the increment of $\pi/2$ if the point passed the crack end.

Thus, the coefficient of the Riemann problem $G(x)$ for infinite contour $(-\infty, \infty)$ has discontinuities at points $x=\pm l$ and satisfies the Hölder condition everywhere except these points. The index is further calculated by summing the index due to rotations of the principal directions on the crack surfaces and the index due to discontinuity of $G$ at $x=\pm l$ (which adds unity).

Therefore the solution of the system should have the form similar to (26) but include a polynomial of $2K$ degree. It is evident that this case is also a special one.

### 4.3 Unit circle

Let $\Gamma$ be the boundary of the unit circle and $\zeta=e^{i\theta}$ be point on it ($-\pi<\theta\leq\pi$). Then the regular operator in (17) with account for single-valuedness (9) becomes
\( \textbf{R} (g', \overline{g'}) = \frac{2a}{\zeta^2}, \quad 2a = \frac{1}{\pi i} \int_{\Gamma} \frac{\text{Re}(g'(t))}{t} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Re}(g'(e^{i\vartheta})) d\vartheta. \) \hspace{1cm} (30)

The constant 2a is real, owing to the fact that \( \text{Re} g' \) is real (the coefficient 2 is introduced for convenience).

The SIE takes the form

\[
\left( e^{i\alpha^+} \sin(\alpha^+ + 2\theta) + e^{i\alpha^-} \sin(\alpha^- + 2\theta) \right) \text{Re} g' + e^{-2i\theta} \sin(\alpha^+ - \alpha^-) \left( 1 \left( \text{Re} g' \right) + 2a \right) = 0
\] \hspace{1cm} (31)

This equation is slightly different from the dominant equation because of the presence of the terms with constant \( a \). However it can be solved in the same way as the dominant equation. By introducing piecewise holomorphic functions \( T^\pm(z) \) that are the boundary values of the Cauchy type integral of the function \( \text{Re} g' \), i.e. \( \text{Re} g' = T^+ - T^- \), \( \text{Im} (\text{Re} g') = T^+ + T^- - 2a \) one transforms (31) to the following homogeneous Riemann boundary value problem

\[ T^+(\zeta) = G(\zeta) T^-(\zeta), \quad \zeta \in \Gamma \] \hspace{1cm} (32)

where the coefficient \( G \) is specified by eqn (20).

This problem has non-trivial solutions if and only if the index of the problem, \( 2K \), is non-negative. The number of homogeneous solutions of the Riemann problem is equal to \( (2K+1) \). However the number of independent solutions of the dominant equation is 2K because of the condition that the Cauchy-type integral vanishes at infinity.

Homogeneous solutions for the Riemann problem can be written in the form (Gakhov [4])

\[ T^+(z) = a + e^{\Lambda^+(z)} P_{2K-1}(z), \quad T^-(z) = a + e^{\Lambda^-(z)} z^{-2K} P_{2K-1}(z) \] \hspace{1cm} (33)

where \( P_{2K-1}(z) \) is a polynomial of order \( 2K-1 \) with arbitrary complex coefficients, \( \Lambda^\pm(z) \) are holomorphic functions determined by the following contour integral

\[ \Lambda^\pm(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln(-t^{-2K} G(t))}{t - z} dt \] \hspace{1cm} (34)

By applying the Sokhotski-Plemelj formulae one finds the boundary values of the piecewise holomorphic function \( \Lambda^\pm(\zeta) \) and further the formal solution of SIE (31) is obtained as the difference \( \text{Re}(g') = T^+ - T^- \). This solution has to satisfy the real valuedness of the function sought, which leads to certain restrictions imposed on the coefficients of the arbitrary polynomials. Details of these calculations are found in [2], where the complete analysis for the interior domain is presented for the case when the second boundary conditions is formulated in terms of normal derivative of principal directions. Here we emphasise that the solution of (31) can only determine \( \text{Re} g' \) while \( \text{Im} g' \) can be chosen arbitrary, which again shows that the complete problem is underspecified.
4.4 Absence of shear stresses

Let us assume that shear stresses on the interface are absent. This means that the following expression is valid

\[ \text{Im} \left( e^{2i\theta(z)} D^\pm(z) \right) = 0, \quad z \in \Gamma \]  

(35)

It is evident that (35) replaces boundary conditions (7) because the interface coincides with a stress trajectory of one orthogonal family. Therefore, the condition \( \alpha^+ \pm \alpha^- \neq 0, \pm \pi \) is not valid and one should consider the system in (12) that is further reduced to a single equation (because non-integral terms vanish while the integral operators are the same in both equations of the system). This results in the following integral equation

\[ \text{Im} \left( \mathbf{I} \left( \text{Re} g' \right) + e^{2i\theta} \mathbf{R} \left( g', g' \right) \right) = 0 \]  

(36)

It can be shown that this equation is a Fredholm-type equation by taking into account that

\[ \mathbf{I} \left( \text{Re} g' \right) = -\mathbf{I} \left( \text{Re} g' \right) + \text{Reg} \left( \text{Re} g' \right) \]  

(37)

where Reg(…) is a regular operator.

Therefore, one arrives at a single integral equation for the determination of two real valued functions. One can separate all terms containing \( \text{Im} g' \) in the right-hand side and all terms containing \( \text{Reg} g' \) in the left-hand side of the equation. The right-hand side can be temporary considered as a known function. Then, in accordance with the Fredholm theorems, this non-homogeneous equation is solvable if and only if the homogeneous one is non-solvable and vice versa. This eventually means that equation (36) is either unsolvable or has an infinite number of solutions because \( \text{Im} g' \) can be chosen arbitrary. Therefore this case also belongs to the degenerate cases.

5 Conclusions

Solvability of the BVP formulated in terms of principal directions given on a contour separating interior and exterior domains has been investigated for the case when the stress vector is continuous across the contour. The index of the corresponding singular integral equation, \( 2K \), has to be non-negative to provide existence of solutions. For any negative index no bounded solutions exist. It has been shown that the solution includes a polynomial of \( 2K \) order, which means that up to \( 4K+2 \) real constants remain to be free parameters of the solution.

Special cases in which the problem is underspecified have been identified; these include adjacent half-planes, a crack in the plane, the circumference of a circle and the case when the interface is free of shear stresses.

References


