Solving Poisson's equations by the Discrete Least Square meshless method

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Abstract

A meshless method is proposed in this paper for the solution of two-dimensional elliptic problems. The proposed method does not require any mesh so it is truly a meshless method. The approach termed generically the "Discrete Least Square meshless method" is applied to discrete the governing differential equations in inner and boundary nodes. A functional is defined as the sum of the squared residual of the governing differential equation and the boundary conditions at the nodal points. Moving least-square (MLS) interpolation is used to construct the shape function's values, which have high continuity in the problem domain. To evaluate the accuracy of the method as an alternative meshless method the development and theory of this new approach is presented in the context of the solution of 2D elliptic equations. Numerical results show that the method possesses high accuracy with low computational effort.
Keywords: meshless method, Discrete Least Square, elliptic problems.

1 Introduction

The idea of using finite difference simplicity and finite element capability of handling complex geometries are the subject of many researches. This is mainly because the mesh generation part of the solution has shown to be a very time consuming challenge especially in finite element applications. The idea of developing methods requiring no mesh has led to the emerging of a new class of the so-called 'meshless' methods. Many of the meshless methods developed so far require background mesh to carry out numerical integration. The integration cells, however, need not be compatible with nodes and thus they can be generated more easily than the FEM meshes. The existing meshless methods can

WIT Transactions on Modelling and Simulation, Vol 42, © 2006 WIT Press
www.witpress.com, ISSN 1743-355X (on-line)
doi:10.2495/BE06003

be generally divided in two main categories depending on the way the discrete equations are formed.

First are the methods based on the weak form of the given differential equation. All these methods use one form of the weighted residual such as Galerkin or Petrov-Galerkin for discretization of the governing differential equations. In this category, one can find Smooth Particle Hydrodynamics (SPH) by Monaghan [1], which is the oldest of the meshless methods, Reproducing Kernel Particle method (RKPM) [2]. These methods use finite integral for function approximation; Partition of Unity (PU) method [3]; hp cloud method [4]; Diffuse Element Method (DEM) by Nayroles et al. [5]; Element Free Galerkin (EFG) by Belytschko et al. [6]. Atluri and Zhu [7] and Zhu et al. [8] suggested the local Petrov-Galerkin and local boundary integral equation (LBIE) approach in which integration is performed locally on each subdomain. A common feature of all these methods is the need for numerical integration requiring a mesh of quadrature points in the domain. Construction of appropriate integration cells, however, is a difficult job and can make meshfree methods less effective. For these reasons, Beissel and Belytschko [9] suggested a nodal integration procedure instead of using Gaussian quadrature in establishing the coefficients of the system of equations. J.X. Zhou et al [10] proposed a nodal integration procedure based on Voronoi diagram for general Galerkin meshless methods. Some of the meshless methods use finite series for function approximation which include Polynomial Point Interpolation Method (PPIM), Radial Point Interpolation Method or Radial Basis Function (RBF) by Chen et al. [11] and well-known Moving Least Square Method (MLS) described by Lancaster and Salkauskas [12] and used by Nayroles et al. [5].

Second are the methods starting directly from the governing equation such as finite point method by Onate et al. [13]. These methods are often arrived at using a point collocation weighted residual formulation of the problem. The collocation method, however, can suffer from the stability problem as that encountered in the nodal integration. In addition, it requires higher-order derivatives of shape functions and results in non-symmetric stiffness matrices.

In this paper we use a fully least square approach in both of the governing differential equation discretization and function approximation which are the main components of every meshless method. The outline of this paper is organized as follows. The moving least square approximation for establishing shape functions is briefly described in section 2. The Discrete Least Square method for discretizing the governing differential equation is presented in section 3. Two elliptic problems solved and the results are presented in section 4. And we close with some concluding remarks in section 5.

2 Moving Least Square (MLS) method

The method of Moving Least Squares (MLS) has been widely used for function approximation by meshless community. The advantages of MLS are three folds: first, there is no need for explicit meshes in the construction of MLS shape functions. Second, high order continuity of shape functions so constructed eliminates the necessity of using weak form of governing equations as required in finite element method (FEM) using standard shape functions. In addition, higher order continuity, if required, is not introduced at the expense of increasing the unknown parameters as usually practiced in FEM. Third; the availability of smooth derivatives eliminates the need for costly procedure of gradient recovery, which is usually required by standard FEM.

In MLS, the function to be approximated is represented by:

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x})\, a_i(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\,\mathbf{a}(\mathbf{x}) \tag{1}$$

Here $\mathbf{p}^T(\mathbf{x})$ is a set of linearly independent polynomial basis and $\mathbf{a}(\mathbf{x})$ represents the unknown coefficients to be determined by the fitting algorithm. The polynomial bases of order m in one and two dimension are given by:

$$\mathbf{p}^T(x) = [1, x, x^2, \ldots, x^m] \tag{2}$$

$$\mathbf{p}^T(\mathbf{x}) = \mathbf{p}^T(x, y) = [1, x, y, x^2, xy, y^2, \ldots, x^m, \ldots, xy^{m-1}, y^m] \tag{3}$$

In the MLS approximation, at each point ($\mathbf{x}$), $\mathbf{a}(\mathbf{x})$ is chosen to minimize the sum of weighted squared residuals defined by:

$$J = \frac{1}{2}\sum_{I=1}^{n} w(|\mathbf{x}\text{-}\mathbf{x}_I|)\left[\mathbf{p}^T(\mathbf{x}_I)\,\mathbf{a}(\mathbf{x}) - u_I\right]^2 \tag{4}$$

Where u_I is nodal value of the function to be approximated, n is the number of nodes and $w(|\mathbf{x}\text{-}\mathbf{x}_I|)$ is the weight function defined to have compact support. The weight functions are chosen to have the following properties:

$$\begin{array}{lll} 1) & w(|\mathbf{x}\text{-}\mathbf{x}_I|) > 0 & \text{On a subdomain} \\ 2) & w(|\mathbf{x}\text{-}\mathbf{x}_I| = 0 & \text{Outside the subdomain} \\ 3) & \int_\Omega w(|\mathbf{x}\text{-}\mathbf{x}_I|)d\Omega = 1 & \text{A normality property} \\ 4) & w(|\mathbf{x}\text{-}\mathbf{x}_I|) & \text{A monotonically decreasing function} \end{array} \tag{5}$$

5) $w(|\mathbf{x}\text{-}\mathbf{x}_I|) \to \delta(s)$ *as* $|\mathbf{x}\text{-}\mathbf{x}_I| = \mathrm{h} \to 0$ where $\delta(\mathrm{s})$, is the Dirac delta function.

Many weight functions are established and used by different researchers. In this paper, we use a cubic spline weight function defined as:

$$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & \text{for} \quad r \le \frac{1}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for} \quad \frac{1}{2} < r \le 1 \\ 0 & \text{for} \quad r > 1 \end{cases} \tag{6}$$

In which, $r = s / s_{\max}$, $s = \left\| \mathbf{x} - \mathbf{x}_I \right\|$ and $s_{\max}$ is the radius of the support. Eqn (4) can be written in matrix form as

$$\mathbf{J} = (\mathbf{Pa} - \mathbf{u})^T \mathbf{W} (\mathbf{Pa} - \mathbf{u}) \tag{7}$$

Where

$$\mathbf{u}^T = (u_1, u_2, \ldots, u_n) \tag{8}$$

$$\mathbf{P} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix} \tag{9}$$

and

$$\mathbf{W(x)} = \begin{bmatrix} w(|\mathbf{x} - \mathbf{x}_1|) & 0 & \ldots & 0 \\ 0 & w(|\mathbf{x} - \mathbf{x}_2|) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & w(|\mathbf{x} - \mathbf{x}_n|) \end{bmatrix} \tag{10}$$

The coefficients **a** are found by minimizing $\mathbf{J}$ with respect to these coefficients. Carrying out the differentiation:

$$\frac{1}{2}\frac{\partial \mathbf{J}}{\partial \mathbf{a}} = \mathbf{A(x)a(x)} - \mathbf{B(x)u} = 0 \tag{11}$$

Where

$$\mathbf{A} = \mathbf{P}^T \mathbf{W(x)P} \tag{12}$$

$$\mathbf{B} = \mathbf{P}^T \mathbf{W(x)} \tag{13}$$

Solving the above equation for the unknown parameters.

$$\mathbf{a(x)} = \mathbf{A}^{-1}\mathbf{(x)B(x)u} \tag{14}$$

The approximation of the unknown function can now be written as

$$\mathbf{u}^h \mathbf{(x)} = \sum_{I=1}^{n} \mathbf{N}_I \mathbf{(x)u}_I \tag{15}$$

where the shape functions are defined as:

$$\mathbf{N} = \mathbf{p}^T \mathbf{(x)A}^{-1}\mathbf{(x)B(x)} \tag{16}$$

In this case, $\mathbf{u}_I \neq \mathbf{u}^h(x_I)$, so the parameters $\mathbf{u}_I$ cannot be treated like nodal values of the unknown function. The shape functions are not strict interpolates since they do not pass through the data. The shape functions do not satisfy the Kronecker delta condition:

$$N_i(\mathbf{x}_j) \neq \delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases} \tag{17}$$

where $N_i(\mathbf{x}_j)$ is the shape function of node i evaluated at node j and δ_{ij} is the Kronecker delta.

3 Discrete Least Square (DLS) method

Consider the following differential equation

$$L(\mathbf{u}) = \mathbf{f} \qquad \text{In } \Omega \tag{18}$$

$$B(\mathbf{u}) = \mathbf{g} \qquad \text{On } \Gamma \tag{19}$$

Where L and B are the differential operator defined on the problem domain Ω and its boundary (Γ), respectively. The philosophy of least square is to find an approximate solution that results in minimum residual error when substituted into equations (18) and (19). The first step is to assume the form of approximate solution ($\mathbf{u}^h$), including a total of m parameters which can be adjusted to minimize the error. This is sometimes called a trial solution, and can be represented by:

$$\mathbf{u}(\mathbf{x}) \cong \mathbf{u}^h(\mathbf{a}, \mathbf{x}) \tag{20}$$

Where $\mathbf{a}$ is the vector of unknown parameters and $\mathbf{x}$ represents the independent variables of the domain. The error is measured by the residuals that result when $\mathbf{u}^h$ is substituted into eqns (19) and (20).

$$\mathbf{R}_\Omega(\mathbf{a}, \mathbf{x}) = L(\mathbf{u}^h) - \mathbf{f} \qquad \text{for } \mathbf{x} \text{ in } \Omega \tag{21}$$

$$\mathbf{R}_\Gamma(\mathbf{a}, \mathbf{x}) = B(\mathbf{u}^h) - \mathbf{g} \qquad \text{for } \mathbf{x} \text{ on } \Gamma \tag{22}$$

Where $\mathbf{R}_\Omega$ and $\mathbf{R}_\Gamma$ are called interior and boundary residuals, respectively. Finally, a weighted sum of squared residuals is minimized over the domain (Ω), establishing the best values of the parameters $\mathbf{a}$. In the Discrete Least Square formulation, the squared residuals are evaluated and summed at the set of points $\mathbf{x}_i$ chosen to represent the problem domain Ω and its boundary (Γ).

$$\mathbf{I}_d(\mathbf{a}) = \frac{1}{2}\sum_{i=1}^{ne}\left[\mathbf{R}_\Omega(\mathbf{a}, \mathbf{x}_i)\right]^2 + \frac{\alpha}{2}\sum_{i=1}^{nb}\left[\mathbf{R}_\Gamma(\mathbf{a}, \mathbf{x}_i)\right]^2 \tag{23}$$

where ne and nb are the number of points chosen on the domain Ω and the boundary Γ, respectively. The factor α in above equation is the relative weight of the boundary residuals with respect to the interior residuals. It is the same of weight coefficient in general penalty method for boundary condition imposing and is equal unit in this paper.

Minimization of the eqn (23) leads to:

$$\frac{\partial \mathbf{I}}{\partial \mathbf{a}} = \sum_{i=1}^{ne}\left[\frac{\partial \mathbf{R}_{\Omega}(\mathbf{a},\mathbf{x}_i)}{\partial \mathbf{a}}\right]\left[\mathbf{R}_{\Omega}(\mathbf{a},\mathbf{x}_i)\right] + \alpha \sum_{i=1}^{nb}\left[\frac{\partial \mathbf{R}_{\Gamma}(\mathbf{a},\mathbf{x}_i)}{\partial \mathbf{a}}\right]\left[\mathbf{R}_{\Gamma}(\mathbf{a},\mathbf{x}_i)\right] = 0 \quad (24)$$

Substitution of $\mathbf{u}^h = \sum_{i=1}^{nn} N_i \mathbf{u}_i = \boldsymbol{Nu}$ in eqns (21), (22) and (24) yields the final system of equations.

$$\boldsymbol{KU=F}$$

System of algebraic equations should be solved for the vector of unknown parameters $\boldsymbol{U}$. Here $nn = ne + nb$ denotes the total number of nodes used to represent the problem domain of it body.

4 Numerical Investigation

In this section, two numerical examples in the area of elliptic problems are solved and results are presented to illustrate the performance of the proposed discrete least square meshless method. We consider two dimensional steady state heat conduction or seepage equation in a homogeneous orthotropic body.

$$\frac{\partial}{\partial x}\left(D\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \phi}{\partial y}\right) = S \quad (25)$$

Subject to appropriate Dirichlete and Newmann boundary conditions

$$\begin{aligned} \phi &= \bar{\phi} && \text{on } \Gamma_{\mathrm{u}} \\ D\frac{\partial \phi}{\partial \mathrm{n}} &= \bar{\mathrm{q}} && \text{on } \Gamma_{\mathrm{q}} \end{aligned} \quad (26)$$

The residuals on interior and boundary nodes are defined by

$$\begin{aligned} \mathbf{R}_{\Omega} &= D\frac{\partial^2 \phi}{\partial x_{\mathrm{j}}^2} - S \neq 0 \qquad \mathrm{j} = 1{,}2 \\ \mathbf{R}_{\Gamma_{\mathrm{u}}} &= \phi - \bar{\phi} \\ \mathbf{R}_{\Gamma_{\mathrm{q}}} &= D\frac{\partial \phi}{\partial x_{\mathrm{j}}} n_{\mathrm{j}} - \bar{\mathrm{q}} \end{aligned} \quad (27)$$

n_{j} is the jth component of the outward unit normal vector to the boundary Γ_{q}.

General differential operators in eqns (18), (19) are defined as.

$$L(\cdot) = \left[D\frac{\partial^2(\cdot)}{\partial x_{\mathrm{j}}^2}\right] \qquad \text{on } \Omega \text{ and } \mathrm{j} = 1{,}2 \quad , \quad \mathbf{f} = S \quad (28)$$

$$\mathbf{B}(.) = 1.0 \quad , \quad \mathbf{g} = \bar{\phi} \quad \text{on } \Gamma_u$$

$$\mathbf{B}(.) = D\frac{\partial(.)}{\partial x_j} n_j \quad , \quad \mathbf{g} = q \quad \text{on } \Gamma_q \tag{29}$$

Application of DLS method leads to the following system of equations

$$\mathbf{k} \ \varphi = \mathbf{f} \tag{30}$$

Where φ is the vector of unknown parameters $\left[\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n\right]^T$.

$$\mathbf{k}_{lm} = \sum_{i=1}^{ne}\left[D\frac{\partial^2 N_l}{\partial x_j^2}\right]_i^T \left[D\frac{\partial^2 N_m}{\partial x_j^2}\right]_i + \sum_{i=1}^{nb}\left[BN_l\right]_i^T \left[BN_l\right]_i \tag{31}$$

$$\mathbf{f}_l = \sum_{i=1}^{ne}\left[D\frac{\partial^2 N_l}{\partial x_j^2}\right]_i^T S_i + \sum_{i=1}^{nb}\left[BN_l\right]_i^T g_i \tag{32}$$

4.1 Poisson equation

Consider the solution of the Poisson's equation.

$$\nabla^2 u(x, y) = \mathbf{sin}\, \pi x \, \mathbf{cos}\, \pi y \qquad \Omega(x, y): \{0 \le x \le 1, \quad 0 \le y \le 1\}$$

Boundary conditions given as

$u = 0 \qquad x = 0$

$u = 0 \qquad x = 1$

$u = 0 \qquad y = 0$

$u = 0 \qquad y = 1$

The exact solution of the governing equation is given by

$$u = \frac{1}{2\pi^2}\mathbf{sin}\, \pi x \, \mathbf{cos}\, \pi y$$

Numerical solutions are obtained on two sets of nodal spacing. First with 121 nodes (11×11) and second with 676 nodes (26×26). Polynomial order is chosen zero order $\mathbf{p} = [x^0 y^0] = [1]$ and subdomain of every node includes one nearest node on both side and both direction (np=1.0, ns=3.0). Figures 1 and 2 show numerical and exact solution of two sections (0.2, 0.5meter) of the problem domain for two nodal distributions.

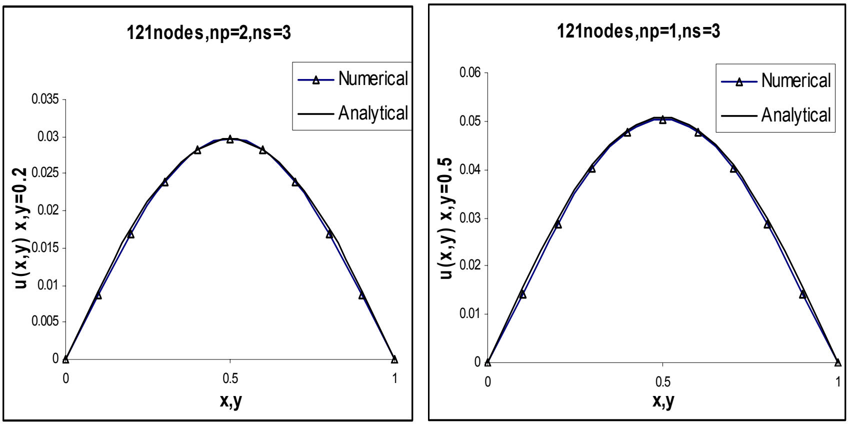

Figure 1: Section plot of Laplace solution with 121nodes.

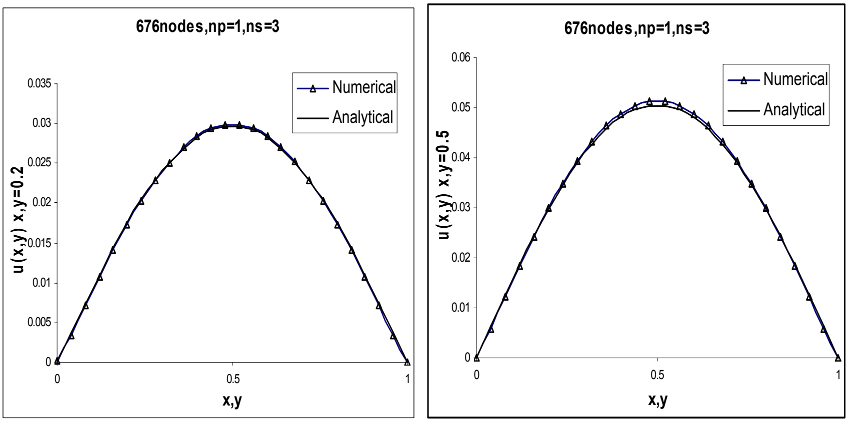

Figure 2: Section plot of Laplace solution with 676 nodes.

4.2 Seepage problem

We consider Seepage problem with this governing equation and Dirichlete and Newmann boundary conditions.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \Omega(x,y) : \begin{Bmatrix} 0 \le x \le 2 \\ 0 \le y \le 1 \end{Bmatrix}$$

Subject to

$\phi = 35.0 \qquad 0 \le x \le 1 \;, \quad y = 1$

$\phi = 0.0 \qquad 1 \le x \le 2 \;, \quad y = 1$

$\frac{\partial \phi}{\partial n} = 0.0 \qquad$ on other boundaries

Problem domain discretizes 3321(41×81) nodes $(\Delta x = \Delta y = 0.025)$. A polynomial of zero order $p = [x^0 y^0] = [1]$ is used (np=1). Same as previous example every subdomain includes two nearest neighbor nodes on both side and both direction (ns=3). Figure 3 shows a countor plot of ϕ results in Problem domain. As shown in figure the distribution of potential are smooth inparticular near the Newmann boundaries.

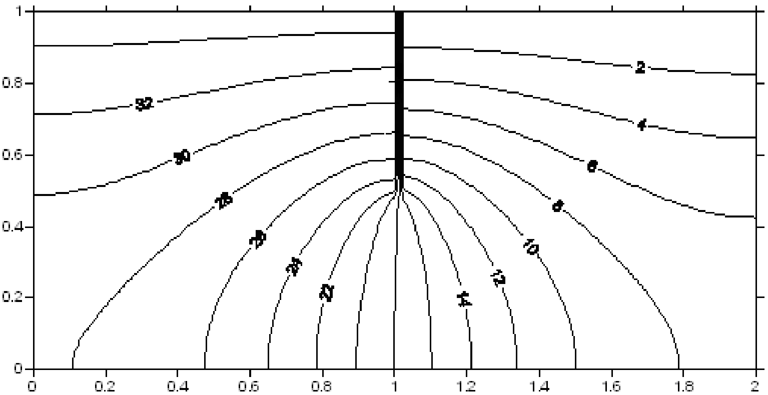

Figure 3: Seepage problem solution (np=1, ns=3).

5 Concluding remarks

In this paper, we present Discrete Least Squares (DLS) meshless method for the solution of elliptic problems. A fully Least Squares method is used in both function approximation and the discretization of the governing differential equations. The meshless shape functions are derived using the Moving Least Squares (MLS) method of function approximation. The discretized equations obtained via a discrete least squares method in which the sums of the squared residuals minimized with respect to unknown nodal parameters. The proposed method has the additional advantages of the producing symmetric, positive definite matrices even for non-self adjoint operators. The method is tested against two elliptic examples in two dimensional steady state forms.

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