On the best recovery of linear functional and its applications

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Abstract
The problem of the best recovery in a sense of Sard of linear functional \( Lf, f \in W_2^q[a,b] \) - Sobolev space, on the basis of information \( T(f) = \{(L_j f), j = 1, 2, \ldots, n\} \) is considered. It is shown that this leads to the best approximation of \( LK \) in the space \( S = \text{span}\{L_j K\}, j = 1, 2, \ldots, n \), where \( K = (x-t)^{q-1}/(q-1)! \) is a truncated power kernel. This problem is solved using Gramm-Schmidt orthogonalization and the best recovery of \( Lf \) is obtained in analytical form. Two applications are considered - interpolation of a function on the basis of given values in some points and of given local mean values. The solutions are given in analytical form which differs from the solutions obtained after solving linear systems. It is shown that the obtained solution of the first problem is in fact a solution of the optimization problem \( \min \|f^{(q)}\|, f(a_i) = f_i, i = 1, 2, \ldots, n \). An algorithm and a program are given using MATLAB-product.

1 Introduction

We consider the Sobolev space

\[
W_2^q[a,b] = \{f \in C_{[a,b]}^{q-1} : f^{(q-1)} \text{ is abs. cont.}, \|f^{(q-1)}\|_{L^2[a,b]} < \infty\},
\]

where

\[
(f,g)_{L^2[a,b]} = \int_a^b f(x)g(x)dx.
\]
We consider functions belonging to the class

\[ B = \{ f \in W_2^q[a, b] : \| f^{(q)} \| \leq 1 \}. \]

Clearly \( B \) is a convex and centrally symmetric body in \( W_2^q[a, b] \).

We are looking for the best approximation of a given functional \( Lf \) on the basis of information

\[ T(f) = (L_1 f, L_2 f, \ldots, L_n f), f \in B, \]

\( L_i \) are linear functionals. In particular, the case

\[ Lf = \int_a^b f(t) dt, \quad L_i f = f(t_i), \quad i = 1, 2, \ldots, n \]

was considered by A. Sard [1], [2], who investigated the kernel of the integral representation of the error. In according with Smolyak’s lemma [3], [4], there exist numbers \( A_1, A_2, \ldots, A_n \) so that the error \( E(T) \) of the best recovery in a sense of Sard can be reached:

\[
E(T) = \inf_{C_i} \sup_{f \in B} \left\| Lf - \sum C_j L_j f \right\| = \sup_{f \in B} \left\| Lf - \sum A_j L_j f \right\|. \quad (1)
\]

2 Best recovery of a linear functional

An arbitrary linear method with a limited error is exact for polynomials of degree \( q - 1 \). Really, if we assume that there exists a linear method of recovery \( Lf \sim \sum C_j L_j f \), which is not exact in \( \pi_{q-1} \), then there exists \( p \in \pi_{q-1} \) such that

\[ Lp - \sum C_j L_j p = \varepsilon \neq 0. \]

But for every \( C \in R \) we have \( C p \in \pi_{q-1} \) and \( L(C p) - \sum C_j L_j (C p) = C \varepsilon \). It means, that the error of this method can be unlimited when \( C \) is arbitrarily large. So, we consider linear methods which are exact in \( \pi_{q-1} \).

**Theorem 1** The error \( E(T) \) and the coefficients \( \{ A_j \} \) of the best linear method of recovery can be obtained by the best linear approximation of \( LK \) in the space \( S = \text{Span}\{ L_j K \}_{j=1}^n \), where \( K \) is a truncated power kernel

\[ K_t(x) = \frac{(x - t)^{q-1}}{(q-1)!} \]

and \( LK, L_j K, j = 1, 2, \ldots, n \) are integrable over \( [a, b] \).

**Proof:** We use the well-known Taylor’s formula with integral form of the remainder

\[ f(x) = p_{q-1}(x) + \frac{1}{(q-1)!} \int_a^b (x - t)^{q-1} f^{(q)}(t) dt. \]
Applying the functionals $L$ and $\sum C_j L_j$ to this equation for a fixed $x$ and using the fact, that the method is exact over $\pi_{q-1}$, we obtain

$$Lf - \sum C_j L_j f = \frac{1}{(q-1)!} \int_a^b \left[ L(x - t)^{q-1} \right] f^{(q)}(t) dt.$$ 

We denote the truncated power kernel by

$$K(x,t) = K_t(x) = \frac{(x-t)^{q-1}}{(q-1)!}.$$ 

Using the Cauchy-Bouniakovski inequality and $f \in B$, the following inequalities are true:

$$|Lf - \sum C_j L_j f| = \left| \int_a^b (LK - \sum C_j L_j K) f^{(q)} \right|$$

$$\leq \left\{ \int_a^b (LK - \sum C_j L_j K)^2 \right\}^{\frac{1}{2}} \left\{ \int_a^b (f^{(q)})^2 \right\}^{\frac{1}{2}}$$

$$= \|LK - \sum C_j L_j K\| \|f^{(q)}\| \leq \|LK - \sum C_j L_j K\|.$$ 

The equality can be reached when

$$f^{(q)}(t) = L(K(x,t)) - \sum C_j L_j K(x,t).$$

The error $E(C)$ by the linear method with coefficients $\{C_j\}_{j=1}^n$ is reached

$$E(C) = \sup_{f \in B} |Lf - \sum C_j L_j f| = \|LK - \sum C_j L_j K\|.$$ 

The error of the best linear method with coefficients $\{A_j\}_{j=1}^n$ is

$$E(T) = \inf_{C_j} \sup_{f \in B} |Lf - \sum C_j L_j f| = \sup_{f \in B} |Lf - \sum A_j L_j f| = \|LK - \sum A_j L_j K\|.$$ 

But

$$E(T) = \inf_{C_j} \|LK - \sum C_j L_j K\|,$$

therefore

$$E(T) = \inf_{C_j} \|LK - \sum C_j L_j K\| = \|LK - \sum A_j L_j K\|.$$ 

The coefficients $\{A_j\}$ can be obtained by the best approximation of $LK$ in the space $S = \text{span}\{L_j K\}_{j=1}^n$ with the same error $E(T)$. 


3 Best approximation of $L_K$ in $\text{span}\{L_j K\}_{j=1}^n$

We are looking for the best approximation of $L_K$ in $S = \text{span}\{L_j K\}_{j=1}^n$ with coefficients $\{A_j\}$. The best approximation is the orthogonal projection of $L_K$ onto $S$. Using Gramm-Schmidt orthogonalization, we can obtain an orthogonal system of functions $\{Q_i K\}$ by the scheme:

$$Q_1 K := L_1 K$$

$$\vdots$$

$$Q_i K = \sum_{j=1}^{i} f_{ij} L_j K,$$

where $(Q_i K, Q_j K) = 0$ for $i \neq j$ and $(Q_i K, Q_i K) \neq 0$. It is known, that

$$Q_i K = \begin{vmatrix}
(L_1, L_1) & (L_1, L_2) & \ldots & (L_1, L_{i-1}) & L_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(L_j, L_1) & (L_j, L_2) & \ldots & (L_j, L_{i-1}) & L_j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(L_i, L_1) & (L_i, L_2) & \ldots & (L_i, L_{i-1}) & L_i
\end{vmatrix}.$$ 

The coefficients $f_{ij}$ in the above equation are [5]

$$f_{ij} = (-1)^{i+j} \begin{vmatrix}
(L_1, L_1) & (L_1, L_2) & \ldots & (L_1, L_{i-1}) \\
\vdots & \vdots & \ddots & \vdots \\
(L_{j-1}, L_1) & (L_{j-1}, L_2) & \ldots & (L_{j-1}, L_{i-1}) \\
(L_j, L_1) & (L_j, L_2) & \ldots & (L_j, L_{i-1}) \\
\vdots & \vdots & \ddots & \vdots \\
(L_i, L_1) & (L_i, L_2) & \ldots & (L_i, L_{i-1})
\end{vmatrix}.$$ 

Here

$$(L_i, L_j) := (L_i K, L_j K) = \int_a^b L_i(K(x,t))L_j(K(x,t))dt.$$ 

The coefficients $\{B_i\}$ must be chosen such that

$$L_K - \sum B_i Q_i, Q_k = 0, k = 1, 2, \ldots, n.$$ 

Since $\{Q_i\}$ is an orthogonal system,

$$B_k = \frac{(L_K, Q_k)}{(Q_k, Q_k)}, k = 1, 2, \ldots, n. \quad (2)$$

Let’s find the connection between $\{A_j\}$ and $\{B_i\}$:

$$\sum_{j=1}^{n} A_j L_j = \sum_{k=1}^{n} B_k Q_k = \sum_{k=1}^{n} \sum_{j=1}^{k} B_k f_{kj} L_j =$$
where

\[ A_j = \sum_{i=j}^{n} B_i f_{ij} = \sum_{i=j}^{n} \frac{(LK, Q_i)}{(Q_i, Q_i)} f_{ij}. \]  

The best linear recovery of \( Lf \) is

\[ Lf \sim \sum_{j=1}^{n} A_j L_j f = \sum_{j=1}^{n} A_j L_j f = \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{(LK, Q_i)}{(Q_i, Q_i)} f_{ij} L_j f. \]

### 4 Best interpolation by given points

Let’s fix \( x \in [a, b] \) and consider the case

\[ Lf := f(x), \quad L_j f := f(a_j), \quad j = 1, 2, \ldots, n, \]

In this case the best recovery is

\[ f(x) \sim \sum_{j=1}^{n} A_j(x) f(a_j) \]

where \( A_j = A_j(x) \) are the unknown functions. Here

\[ L_j(K(x, t)) = \frac{(a_j - t)^{q-1}}{(q-1)!}. \]

The best approximation of \( LK \) in \( S = \text{span}\{(a_j - t)^{q-1}\}_{j=1}^{n} \) can be found:

\[ Q_i = \sum_{j=1}^{i} f_{ij}(a_j - t)^{q-1}, \quad B_i = B_i(x) = \frac{(LK, Q_i)}{(Q_i, Q_i)}. \]

\[ (LK, Q_i) = \int_{a}^{b} \frac{(x-t)^{q-1}}{(q-1)!} \sum_{j=1}^{i} f_{ij}(a_j - t)^{q-1} \frac{1}{(q-1)!} dt = \frac{1}{[(q-1)!]^2} \sum_{j=1}^{i} f_{ij} I_j(x). \]

Here

\[ I_j(x) = \int_{a}^{b} (x-t)^{q-1}(a_j - t)^{q-1} dt. \]

We will show that

\[ I_j(x) = \sum_{k=0}^{q-1} \frac{(q-1)}{2q-k-1} (m_j(x) - a)^{2q-k-1} |a_j - x|^k, \]  

(5)
where \( m_j(x) = \min(a_j, x) \).

Really, by the above symbol

\[
I_j(x) = \int_a^{m_j(x)} (x - t)^{q-1}(a_j - t)^{q-1} dt
\]

and after the substitution \( y = m_j(x) - t \) it follows

\[
I_j(x) = \int_0^{m_j(x)-a} y^{q-1}|y + |a_j - x| |^{q-1} dy =
\]

\[
= \sum_{i=0}^{q-1} \binom{q-1}{i} |a_j - x|^i \int_0^{m_j(x)-a} y^{2q-i-2} dy =
\]

\[
= \sum_{i=0}^{q-1} \binom{q-1}{i} |a_j - x|^i \frac{(m_j(x) - a)^{2q-i-1}}{2q - i - 1}.
\]

Clearly that \( I_j(a_i) = I_i(a_j) \). If \( x < a_j \), then \( m_j(x) = x \) and \( I_j(x) \) is a polynomial of degree \( 2q - 1 \). If \( a_j < x \), then \( m_j = a_j \) and \( I_j(x) \) is a polynomial of degree \( q - 1 \). For \( x = a_j \) follows \( I_j(a_j) = \frac{1}{2q-1} (a_j - a)^{2q-1} \) and it is easy to see that \( I_j(x) \) is continuous over \([a, b]\). Further,

\[
(Q_i, Q_j) = \sum_{r=1}^i \sum_{s=1}^j f_{ir} f_{js}(L_r K, L_s K).
\]

But

\[
(L_r K, L_s K) = \frac{1}{(q-1)!^2} \int_a^b (a_r - t)^{q-1} (a_s - t)^{q-1} dt = \frac{1}{(q-1)!^2} I_r(a_s).
\]

Replacing the above equalities in (2) and (4), we obtain for the best recovery

\[
f(x) \sim \sum_{j=1}^n f(a_j) \sum_{i=j}^n f_{ij} \frac{\sum_{k=1}^i \sum_{s=1}^j f_{ik} I_k(x)}{\sum_{r=1}^n \sum_{s=1}^j f_{is} f_{ir} I_r(a_s)}.
\] (6)

Schoeneberg has been proved [6], that the best approximation in a sense of Sard for \( Lf \) by given information \( \{L_j f = f(a_j)\}_{j=1}^n \) is \( Ls_f \), where \( s_f \) is a natural spline of degree \( 2q - 1 \) with knots - the same points \( a_1, \ldots, a_n \).

Really, in the case \( Lf = f(x) \) it is easy to check, that it’s best recovery - the right side of (6) - is a natural spline \( s_f \) with knots in \( \{a_j\} \) which interpolates them with values \( f(a_j) \).

**Theorem 2** The best recovery for \( f(x) \) in a sense of Sard by given information \( T(f) = \{f(a_1), \ldots, f(a_n)\} \)

\[
f(x) \sim \sum A_j(x) f(a_j)
\]
is a natural spline, interpolating \( \{(a_i, f(a_i))\} \), where \( A_i(a_k) = \delta_{ik}, k = 1, 2, \ldots, n, j = 1, 2, \ldots, n \).

It is known \([4], [7]\), that among all \( g \in W^q_2[a, b] \) interpolating \( (a_i, f(a_i)) \) the natural interpolating spline has a minimal norm of the q-derivative. The solution can be found solving a system of linear equations \([4], [7]\).

We obtain the solution in analytical form from the viewpoint of the best recovery of a functional:

**Theorem 3** The problem

\[
\min_{f(a_i)=f_i} \| f^{(q)} \|
\]

has unique solution – the natural spline, given by equation (6), which is the best recovery of \( f(x) \).

A program is realized which calculates the value in arbitrary point on the basis of given points and a visualization of the results is shown, too. The language of MATLAB - product is used.

**Table 1:** Number of given points \( N=10, q=3, \) degree of spline 5, interval \([20, 290]\), calculated values in points with step 10.

<table>
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<tr>
<th>Data:</th>
<th>20</th>
<th>50</th>
<th>70</th>
<th>100</th>
<th>120</th>
<th>150</th>
<th>180</th>
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<td></td>
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<td></td>
</tr>
<tr>
<td>y</td>
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<td>100</td>
<td>140</td>
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<td>220</td>
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<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td></td>
</tr>
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<td>46.98</td>
<td>70.00</td>
<td>109.36</td>
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<td>110</td>
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<td>130</td>
<td>140</td>
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<td>200</td>
<td>210</td>
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</tbody>
</table>

**5 Interpolation by given local mean values**

Let \( a < a_1 < b_1 < \ldots < a_n < b_n < b \) and let the local mean values are given

\[
L_j f = \int_{a_j}^{b_j} f(x)dx, \quad L f(x) = f(x).
\]

We calculate:

\[
L_j K = \frac{1}{(q-1)!} \int_{a_j}^{b_j} (x-t)^q_+ dx = \frac{(b_j-t)^q_+}{q!}
\]
\[ (Q_i K, Q_i K) = \sum_{r=1}^{i} \sum_{s=1}^{i} f_{ir} f_{is} (L_r K, L_s K) \]
\[ = \frac{1}{q!^2} \sum_{r=1}^{b} \sum_{s=1}^{i} f_{ir} f_{is} \int_{a}^{b} (b_r - t)^{q} (b_s - t)^{q} dt. \]

If we denote
\[ I(b_r^q, b_s^q) = \int_{a}^{b} (b_r - t)^{q} (b_s - t)^{q} dt, \]
by analogy with the integral ( ) follows
\[ I(b_r^q, b_s^q) = \frac{1}{q!^2} \sum_{i=0}^{q} \binom{q}{i} |b_s - b_r|^i \frac{(\min(b_r, b_s) - a)^{2q-i+1}}{2q-i+1}. \]

\[ (LK, Q_i) = \frac{1}{(q-1)!q!} \sum_{j=1}^{i} f_{ij} \int_{a}^{b} (x-t)^{q-1} (b_j - t)^{q} dt. \]

We denote the last integral by \( I(x^{q-1}, b_j^q) \) and calculate it:
\[ I(x^{q-1}, b_j^q) = \int_{a}^{b} (x-t)^{q-1} (b_j - t)^{q} dt = \int_{a}^{n_j} (x-t)^{q-1} (b_j - t)^{q} dt, \]
where \( n_j(x) = \min(x, b_j) \). After the substitution \( y = n_j(x) - t \) we consider two cases:

1) \( n_j(x) = x \).
\[ I(x^{q-1}, b_j^q) = \int_{0}^{x-a} y^{q-1} \sum_{l=0}^{q} \binom{q}{l} (b_j - x)^l y^{q-l} dy \]
\[ = \sum_{l=0}^{q} \binom{q}{l} (b_j - x)^l \frac{(x-a)^{2q-l}}{2q-l}. \]

2) \( n_j(x) = b_j \).
Then \( y = b_j - t \) and
\[ I(x^{q-1}, b_j^q) = \int_{0}^{b_j-a} y^n \sum_{l=0}^{q-1} \binom{q-1}{l} y^{q-l-1} (x-b_j)^l \]
\[ = \sum_{l=0}^{q-1} \binom{q-1}{l} (x-b_j)^l \frac{(b_j-a)^{2q-l}}{2q-l}. \]
Combining these two cases, 

\[ I(x^{q-1}, b_j^q) = \sum_{l=0}^{q-\varepsilon_j} \binom{q-\varepsilon_j}{l} \frac{|x - b_j|}{2q-l} (n_j(x) - a)^{2q-l}, \]

where \( n_j(x) = \min(b_j, x) \) and 

\[ \varepsilon_j = \varepsilon_j(x) = \begin{cases} 
1 & n_j(x) = b_j \\
0 & n_j(x) = x 
\end{cases} \]

Having in mind the representation (4) we obtain 

\[ Lf \approx \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{\sum_{k=1}^{i} f_{ik}I(x^{q-1}, b_j^q)}{\sum_{r=1}^{i} \sum_{s=1}^{i} f_{ir}f_{is}I(b_r^q, b_s^q)} \]

References


