Implementation of an advanced symmetric Galerkin BEM-formulation for 3D-elastostatics

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Abstract

Like the finite element method (FEM), the symmetric Galerkin boundary element method (SGBEM) produces symmetric "stiffness"-matrices, which allow an efficient coupling of both numerical methods. Compared to the standard collocation method, in the SGBEM one has to deal with double integrations and the occurrence of hypersingular kernel functions. For two-dimensional problems the SGBEM is well developed (e.g. [4, 5, 6]), whereas 3D-implementations were restricted to linear triangular elements up to now. This is not sufficient for practical applications in engineering. Based on a strongly singular formulation of the traction boundary integral equation, the regularization procedure for linear triangular elements proposed in [1, 7] is extended to higher order shape functions and quadrilateral elements.

1 Introduction

In view of an efficient coupling of the finite element method and the 3D-boundary element method for elastostatics, a symmetric BEM-formulation is advantageous. Apart from this, a symmetric implementation of the Galerkin-boundary element method (SGBEM) has some more benefits over the standard collocation method:

1. the higher convergence-rate of the Galerkin method [9],
2. the savings in memory, as only half of the system matrix has to be stored,
3. like in the FEM, efficient solvers which exploit the symmetry of the matrix can be used,
4. an easy way to regularize the hypersingular traction boundary integral equation, which offers new possibilities to the DUAL-BEM for crack problems [2],
5. the evasion of difficulties arising at edges and corners.

Up to now, implementations of the 3D-SGBEM were restricted to linear triangular elements [1, 7]. Sauter and Schwab [8] presented an extension to linear quadrilateral elements, which unfortunately uses different transformations to regularize the singular equations.

It will be shown that integration formulas for linear triangular elements can also be applied to arbitrary element types. Thus, the same transformations can be used for triangles and for quadrilaterals, which makes it easy to mix different types of elements in one numerical model.

2 Symmetric 3D Galerkin Formulation

In linear elastostatics, the strongly singular displacement boundary integral equation (u-BIE)

\[ c_{ij}(\xi)u_j(\xi) + \int_{\Gamma} T_{ij}(\xi, x)u_j(x)d\Gamma(x) - \int_{\Gamma} U_{ij}(\xi, x)t_j(x)d\Gamma(x) = 0 \]  

(1)

and the hypersingular traction boundary integral equation (t-BIE)

\[ \bar{c}_{aj}(\xi)t_j(\xi) + \int_{\Gamma} \bar{T}_{aj}(\xi, x)u_j(x)d\Gamma(x) - \int_{\Gamma} \bar{U}_{aj}(\xi, x)t_j(x)d\Gamma(x) = 0 \]  

(2)

are valid, when body forces are neglected. \( \Gamma = \Gamma^u \cup \Gamma^t \) denotes the boundary of the considered body with Dirichlet boundary conditions on \( \Gamma^u \) and Neumann boundary conditions on \( \Gamma^t \). \( u_j \) and \( t_j \) are the cartesian components of the displacement and traction vector, respectively. \( U_{ij}(\xi, x) = O\left(\frac{1}{r}\right) \), \( T_{ij}(\xi, x) = O\left(\frac{1}{r^2}\right) \), \( U_{aj}(\xi, x) = O\left(\frac{1}{r^3}\right) \) and \( \bar{T}_{aj}(\xi, x) = O\left(\frac{1}{r^3}\right) \) denote the singular Kelvin fundamental solutions, depending only on the distance \( r \) between the field point \( x \) and the source point \( \xi \) [1]. The free terms \( c_{ij}(\xi) \) and \( \bar{c}_{aj}(\xi) \) depend on the geometry at the source point \( \xi \). For smooth boundary, \( c_{ij} = \frac{1}{2} \delta_{ij} \)
and $\tilde{c}_{aj} = \frac{1}{2} \delta_{aj}$ hold. After application of a rigid-body movement, eqn. (2) can also be written in the strongly singular form

$$\tilde{c}_{aj}(\xi) t_j(\xi) + \int_{\Gamma} T_{aj}(\xi, x) \left[u_j(x) - u_j(\xi)\right] d\Gamma(x)$$

$$- \int_{\Gamma} \tilde{U}_{aj}(\xi, x) t_j(x) d\Gamma(x) = 0. \quad (3)$$

In order to solve the boundary integral equations numerically, the boundary is discretized and the boundary values are approximated via shape functions $N^{lm}$ [3]. Equations (1) and (3) can be rewritten as:

$$c_{ij}(\xi) u_j(\xi) + \sum_{l=1}^{N} \sum_{m=1}^{q} u_j^{lm} \int_{\Gamma_l} T_{ij}(\xi, x(\eta)) N^{lm}(\eta) d\Gamma^l$$

$$- \sum_{l=1}^{N} \sum_{m=1}^{q} t_j^{lm} \int_{\Gamma_l} U_{ij}(\xi, x(\eta)) N^{lm}(\eta) d\Gamma^l \quad (4)$$

and

$$\tilde{c}_{aj}(\xi) t_j(\xi) + \sum_{l=1}^{N} \sum_{m=1}^{q} \int_{\Gamma_l} \tilde{T}_{aj}(\xi, x(\eta)) \left[N^{lm}(\eta) u_j^{lm} - u_j(\xi)\right] d\Gamma^l$$

$$- \sum_{l=1}^{N} \sum_{m=1}^{q} t_j^{lm} \int_{\Gamma_l} \tilde{U}_{aj}(\xi, x(\eta)) N^{lm}(\eta_2) d\Gamma^l, \quad (5)$$

where $\eta = (\eta_1, \eta_2)$ denote the local coordinates of the element containing the field point $x = x(\eta)$ and $d\Gamma^l = |J^l(\eta)| d\eta_1 d\eta_2$ with the Jacobian $|J^l(\eta)|$ of the coordinate transformation from the global coordinate system to the local element coordinate system.

Analysing the Kelvin fundamental solutions, one obtains the following symmetry properties:

$$U_{ij}(\xi, x) = U_{ji}(x, \xi) = U_{ij}^T(\xi, x), \quad (6)$$

$$\tilde{T}_{aj}(\xi, x) = \tilde{T}_{ja}(x, \xi) = \tilde{T}_{aj}^T(\xi, x), \quad (7)$$

$$\tilde{U}_{ij}(\xi, x) = T_{ji}(x, \xi) = T_{ij}^T(\xi, x). \quad (8)$$

Two facts prevent the standard collocation method from producing symmetric matrices:

- The integration is carried out only with respect to the field point $x$ and not to the source point $\xi$. Thus, the Jacobian depends only on the element containing the field point $x$. 

In practical problems there are different types of boundary conditions along the surface \( \Gamma \), so integrals over the symmetric \( U_{ij} \)-kernel as well as over the non symmetric \( T_{ij} \)-kernel occupy the matrix.

In order to treat the source point in the same way as the field point, an outer integration with respect to the source point is necessary. This leads to the equations of the Galerkin formulation, obtained by the weighted residual method. Using the shape functions \( N^e(\xi) \) at the source point as weight functions and integrating over the whole boundary with respect to the source point \( \xi \), equations (4) and (5) can be rewritten as

\[
\frac{1}{2} \sum_{e=1}^{E} \sum_{n=1}^{p} u_{i}^{en} \int_{\Gamma^e} N^e(\xi) N^e(\xi) d\Gamma^e \\
+ \sum_{e=1}^{E} \sum_{l=1}^{N} \sum_{m=1}^{q} u_{j}^{lm} \int_{\Gamma^e \Gamma^l} T_{ij}(\xi, \eta) N^e(\xi) N^{lm}(\eta) d\Gamma^l d\Gamma^e \\
- \sum_{e=1}^{E} \sum_{l=1}^{N} \sum_{m=1}^{q} t_{j}^{lm} \int_{\Gamma^e \Gamma^l} U_{ij}(\xi, \eta) N^e(\xi) N^{lm}(\eta) d\Gamma^l d\Gamma^e = 0, \tag{9}
\]

\[
\frac{1}{2} \sum_{e=1}^{E} \sum_{n=1}^{p} t_{a}^{em} \int_{\Gamma^e} N^e(\xi) N^e(\xi) d\Gamma^e + \\
\sum_{e=1}^{E} \sum_{l=1}^{N} \int_{\Gamma^e \Gamma^l} \bar{T}_{aj}(\xi, \eta) N^e(\xi) \left[ \sum_{m=1}^{q} N^{lm}(\eta) u_{j}^{lm} - \sum_{n=1}^{p} N^e(\xi) u_{j}^{en} \right] d\Gamma^l d\Gamma^e \\
- \sum_{e=1}^{E} \sum_{l=1}^{N} \sum_{m=1}^{q} \bar{t}_{j}^{lm} \int_{\Gamma^e \Gamma^l} \bar{U}_{aj}(\xi, \eta) N^e(\xi) N^{lm}(\eta) d\Gamma^l d\Gamma^e = 0. \tag{10}
\]

\( E \) is the set of elements containing the “collocation”-node with the corresponding weight functions \( N^e(\xi) \). \( d\Gamma^e \) denotes \( |J^e(\xi)| d\zeta_1 d\zeta_2 \) with the Jacobian \( |J^e(\xi)| \) of the Galerkin-element \( e \) with its local coordinates \( \xi = (\zeta_1, \zeta_2) \). Only at element edges the free terms \( c_{ij}(\xi) \) and \( \bar{c}_{aj}(\xi) \) may differ from \( \frac{1}{2} \delta_{ij} \). As they are finite and only defined on sets of measure zero, they provide no contribution when performing the outer integration over the elements containing the field point.

In order to exploit the symmetry properties (6) – (8), the boundary value problem at the “collocation”-node must be taken into account. Thus, on \( \Gamma^u \), where Dirichlet boundary conditions are given.
the u-BIE and on \( \Gamma^t \) with Neumann boundary conditions the t-BIE is applied.

Evaluation of equations (9) and (10) on every "collocation"-node depending on the given boundary problem (lower index \( u \) resp. \( t \)) at this node, leads to a symmetric system of equations

\[
\begin{bmatrix}
U_u & T_u \\
\bar{U}_t & \bar{T}_t
\end{bmatrix}
\begin{bmatrix}
x^t \\
x^u
\end{bmatrix} =
\begin{bmatrix}
R_u \\
R_t
\end{bmatrix}
\]

for the calculation of the unknown tractions \( x^t \) and the unknown displacements \( x^u \).

Eliminating the unknown tractions by applying Schur’s complement [7] leads to the FEM-like formulation \( Kx^u = R^* \). In a postprocessing step, the unknown tractions \( x^t \) can be calculated.

3 Evaluation of the double integrals

Due to the double integration with respect to the source point \( \xi \) and the field point \( x \), four typical element constellations have to be considered:

![Relative positions of elements](image)

Figure 1: Relative positions of elements

The regular integration can be carried out using standard Gaussian quadrature formulas. In practical problems, one has to evaluate significantly more regular than singular integrations. Therefore, the regular case is time-dominating and has to be implemented as efficient as possible.

For the numerical evaluation of the singular integrals, a regularization procedure is required. Referring to [1, 7], special transformations are used, which consist of the steps:

- introduction of relative coordinates to move the singularity to the origin of the new coordinate system,
• parameterizing the four-dimensional integration domains in a way, that the integration over the relative coordinate is the outer integration,
• utilization of the symmetry conditions of the kernel functions,
• introduction of Duffy’s triangular coordinates to remove the remaining weak singularities.

As an example, the principle of the procedure will be demonstrated for coincident linear triangular elements:

Generally, one has to deal with integrals of the form:

$$\lim_{\eta \to \zeta} \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_{ij}(\zeta, \eta) N_1^\xi(\zeta) \Phi^m(\eta, \zeta) |J(\zeta)| |J(\eta)| d\eta_2 d\eta_1 d\zeta_2 d\zeta_1$$

(12)

with the weakly, strongly or hypersingular kernel function $K_{ij}$ and

$$\Phi^m(\eta, \zeta) = \begin{cases} N^m(\eta) & \text{(weakly and strongly singular)} \\ N^m(\eta) - N^m(\zeta) & \text{(hypersingular)} \end{cases}$$

(13)

First, we introduce relative coordinates

$$u_1 = \eta_1 - \zeta_1, \quad u_2 = \eta_2 - \zeta_2$$

(14)

in a way that the singularity lies in the origin of the new coordinate system at $u_1 = u_2 = 0$. This leads to new integration boundaries

$$\lim_{\mathbf{u} \to 0} \int_0^1 \int_0^1 \int_{\zeta_1}^{\zeta_1} \int_{-\zeta_2}^{\zeta_2} K_{ij}(\zeta, \mathbf{u} + \zeta) \cdots d\zeta_2 d\zeta_1 d\zeta_2 d\zeta_1.$$
This four-dimensional integration domain can now be split into three pairs of symmetric subdomains. Combination of the first two of the symmetric subdomains in addition to the symmetry properties of the kernel functions

\begin{align*}
U_{ij}(\xi, x) &= U_{ij}(x, \xi), \quad (16) \\
\bar{T}_{aj}(\xi, x) &= \bar{T}_{aj}(x, \xi), \quad (17) \\
T_{ij}(\xi, x) &= -T_{ij}(x, \xi), \quad \bar{U}_{aj}(\xi, x) = -\bar{U}_{aj}(x, \xi) \quad (18)
\end{align*}

leads to

\begin{align*}
&\lim_{u \to 0} \int_{0}^{1} \int_{0}^{1-u} \int_{\zeta_1}^{\zeta_1} K_{ij}(\zeta, u + \zeta) \left[ N^x(\zeta) \Phi^m(u + \zeta, \zeta) \right] \\
&\quad\pm N^x(u + \zeta, \zeta) \Phi^m(\zeta) \left| J(\zeta) \right| |J(u + \zeta)| d\zeta_2 d\zeta_1 d\zeta_2 d\zeta_1. \quad (19)
\end{align*}

The plus-sign holds for the weakly and hypersingular, the minus-sign for the strongly singular kernels.

After introduction of Duffy’s triangular coordinates

\begin{align*}
u_1 &= w_1; \quad u_2 = w_1 w_2; \quad du_1 du_2 = w_1 dw_1 dw_2, \quad (20)
\end{align*}

the distance between source- and field point can be written as

\begin{align*}
r_i &= w_1 R_i(w_1, w_2); \quad r = w_1 \sqrt{R_k R_k}; \quad r_i = \frac{\psi_1 R_i}{\psi_1 \sqrt{R_k R_k}} \quad (21)
\end{align*}

with the regular residual \( R_i = R_i(w_1, w_2) \neq 0 \), independent of the polynomial degree of the used shape functions. Thus, weakly singular kernel functions of order \( O\left(\frac{1}{w_1 \sqrt{R_k R_k}}\right) \) are regularized by the Jacobian \( w_1 \) of Duffy-coordinates. Calculating the difference of shape functions for the strongly singular kernels, one obtains

\begin{align*}
N^x(\zeta) N^m(w + \zeta) - N^x(w + \zeta) N^m(\zeta) = w_1 N^*(w, \zeta), \quad (22)
\end{align*}

which is in addition to the Jacobian \( w_1 \) sufficient to regularize kernel functions of order \( O\left(\frac{1}{w_1^2}\right) \). For the hypersingular kernel together with the rigid-body movement, the sum of shape functions can be calculated as

\footnote{Note that (17) and (18) are only valid for linear triangular elements with a constant normal vector within the element.}
\[ N^\xi(\zeta) [N^m(w + \zeta) - N^m(\zeta)] + N^\xi(w + \zeta) [N^m(\zeta) - N^m(w + \zeta)] \]
\[ = w_1^2 \hat{N}^*(w, \zeta). \]  

Taking the \( w_1 \)-Jacobian into account, the remaining integral is now regular.

Applying one final coordinate transformation only to have constant integration boundaries for the numerical integration and using similar transformations for the remaining two pairs of symmetric subdomains [1], the following now regular expression

\[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left\{ (K_{ij}N^*) \left[ \tilde{\zeta}_1(1 - w_1), \tilde{\zeta}_1 \tilde{\zeta}_2(1 - w_1), \right. \right. \]
\[ w_1 + \tilde{\zeta}_1(1 - w_1), w_1 w_2 + \tilde{\zeta}_1 \tilde{\zeta}_2(1 - w_1) \left. \right] + \]
\[ (K_{ij}N^*) \left[ \tilde{\zeta}_1(1 - w_1) + w_1(1 - w_2), \right. \right. \]
\[ \tilde{\zeta}_1 \tilde{\zeta}_2(1 - w_1) + w_1(1 - w_2), w_1 w_2, w_1(w_2 - 1) \left. \right] + \]
\[ (K_{ij}N^*) \left[ \tilde{\zeta}_1(1 - w_1) + w_1(1 - w_2), \tilde{\zeta}_1 \tilde{\zeta}_2(1 - w_1), \right. \right. \]
\[ w_1 w_2, w_1 \left. \right] \right\} w_1 \tilde{\zeta}_1(1 - w_1)^2 d\tilde{\zeta}_2 d\tilde{\zeta}_1 dw_2 dw_1 \]  

with

\[ (K_{ij}N^*) [\alpha_1, \alpha_2, \beta_1, \beta_2] = \]
\[ K_{ij}(\alpha_1, \alpha_2, \beta_1, \beta_2) \left[ N^\xi(\alpha_1, \alpha_2) \Phi^m(\beta_1, \beta_2, \alpha_1, \alpha_2) \right. \right. \]
\[ \pm N^\xi(\beta_1, \beta_2) \Phi^m(\alpha_1, \alpha_2, \beta_1, \beta_2) \right] |J(\alpha_1, \alpha_2)||J(\beta_1, \beta_2)|, \]

which is suitable for numerical integration via Gaussian quadrature formulas is obtained.

For the edge- and vertex-adjacent cases similar transformations can be used after choosing appropriate coordinate systems. They also lead to regular integration formulas [1, 7]. Note that for these cases the symmetry properties (16) – (18) are not necessary for the regularization procedure.
4 Application to higher order elements

In order to use the same integration formulas for quadrilateral and for triangular elements, which is of advantage when mixing these element types in one model, a subdivision of the quadrilaterals into two triangles only in the singular integration cases will be applied.

A problem arises in the case of coincident elements with shape functions of higher order. As the normal vector is not constant within the element, symmetry properties (17) and (18) no longer hold.

But evaluating the normal vectors $n_i$ at the field point and $\bar{n}_i$ at the source point in a Taylor series at the corresponding other point

$$n_i(\eta) = \bar{n}_i(\zeta) + \bar{O}_i(r); \quad \bar{n}_i(\zeta) = n_i(\eta) + O_i(r),$$  \hspace{1cm} (25)

leads to

$$T_{ij}(x, \xi) = -T_{ij}(\xi, x) + O \left(\frac{1}{r}\right),$$  \hspace{1cm} (26)

$$\bar{U}_{aj}(x, \xi) = -\bar{U}_{aj}(\xi, x) + O \left(\frac{1}{r}\right).$$  \hspace{1cm} (27)

The first terms on the right side of eqns. (26, 27) fulfill the symmetry requirements for the regularization process, whereas the remaining parts are of order $O \left(\frac{1}{r}\right)$. This allows to treat the remaining parts in the same way as the weakly singular kernel function, which can be regularized by applying Duffy’s triangular coordinates.

Substitution of the normal vectors of the hypersingular kernel function leads to

$$\bar{T}_{aj}(x, \xi) = \bar{T}_{aj}(\xi, x) + O \left(\frac{1}{r^2}\right).$$  \hspace{1cm} (28)

The remaining part has to be split in a term, which can be regularized like the strongly singular kernel functions, and a remaining weakly singular term.
References


