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The method of fundamental solutions and compactly supported radial basis functions: a meshless approach to 3D problems

C.S. Chen[1], M.D. Marcozzi[1], S. Choi[2]
[1]*Department of Mathematical Sciences, University of Nevada, Las Vegas, NV 89154, U.S.A.*
E-Mail: chen@nevada.edu, marcozzi@nevada.edu
[2]*Department of Finance, University of Nevada, Las Vegas, NV 89154, U.S.A.*
E-Mail: schoi@ccmail.nevada.edu

Abstract

A simple mesh/grid free numerical method for Poisson's equation has been developed to solve large scale problems in three-dimensions. In this paper, the Poisson's equation is split into two parts by using the method of particular solutions; the particular solution being approximated by the dual reciprocity method (DRM). In the DRM, we choose compactly supported radial basis functions for the interpolation of the forcing term. To then achieve a true meshless method, we employ the method of fundamental solutions (MFS) in order to solve the resulting homogeneouse quation. The approach developed here is especially attractive for complex-shaped boundaries in arbitrary dimensions.

1 Introduction

In the past decade, the application of the Dual Reciprocity method (DRM) [10, 12] in the boundary element literature has grown at a rapid pace, due in large part to its unique ability to alleviate the domain integration which occurs when the governing differential equation contains a nonhomogeneous term. The recent theoretical development [6, 8] of the DRM using radial basis functions (RBFs) has put the DRM on a firm mathematical foundation. As a result, the DRM has been confirmed as a reliable numerical method in solving various kind of partial differential equations. Recently, the discovery of analytic particular solutions for Helmholtz operator using thin plate splines and high-order splines [9] have further extended the applicability of the DRM. In general, RBFs are globally defined basis functions. In the process of interpolating nonhomogeneous term, however, they lead to a dense matrix which may become ill-conditioned for large number of interpolation points, especially in the higher dimensional cases. To overcome this difficulty, we adopt compactly supported radial basis functions (CS-RBFs), which have been recently developed by Wu [15] and Wendlend [14] for the purpose of interpolating large scale problems. We refer the reader to Schaback [13] for an excellent review of CS-RBFs. In the context of the DRM, CS-RBFs have been implemented in a simple 2D case [3]. As the choice of interpolation nodes for the classical RBFs and CS-RBFs are usually quite arbitrary, we use quasi-random points [11] to ensure the uniform distribution of collocation points. This is the first part of our so-called meshless approach for the evaluation of particular solutions.

The second part of our meshless scheme is to employ the method of fundamental solutions (MFS) for the solution of elliptic boundary value problems. In some sense, the MFS and the boundary element method are very similar. Both methods require a known fundamental solution of the govenering equation. One of the advantages of the BEM is its ability to reduce a problem such that only boundary discretization is required. However, it still remains a formidable task to discretize the surface of an irregular domain in three-dimensions [1]. Alternately, the MFS may be regarded as a meshless method for solving homogeneous differential equations. The MFS requires neither domain nor boundary discretization. Furthermore, boundary integration is not required and so the MFS is very efficient in terms of numerical computation. The MFS is therefore especially attractive in higher dimensional cases. Recently, there are two excellent review papers [4, 7] devoted to this subject. In this paper, we coupled the complementary MFS and the DRM using CS-RBFs to achieve a complete meshless method. A numerical example in three-dimensions is given to demonstrate the effectiveness of our method.

2 The Method of Particular Solutions

Here we consider Poisson's equation with Dirichlet boundary conditions;

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbf{\Omega}, \tag{1}$$
$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\mathbf{\Omega}, \tag{2}$$

where $\mathbf{\Omega} \subset \mathbb{R}^3$ is a bounded open nonempty domain with sufficiently regular boundary $\partial\mathbf{\Omega}$. It is well-known that the nonhomogeneous term in (1) can be eliminated by the use of a particular solution. Let $v = u - u_p$, where u_p is a particular solution satisfying the nonhomogeneous equation

$$\Delta u_p(\mathbf{x}) = f(\mathbf{x}) \tag{3}$$

but does not necessary satisfy the boundary condition in (2). The function v then satisfies the homogeneous boundary value problem

$$\begin{aligned} \Delta v(\mathbf{x}) &= 0, \quad \mathbf{x} \in \boldsymbol{\Omega}, & (4)\\ v(\mathbf{x}) &= g(\mathbf{x}) - u_p(\mathbf{x}), \quad \mathbf{x} \in \partial\boldsymbol{\Omega}. & (5) \end{aligned}$$

Since v satisfies Laplace's equation, the integral formulation for v does not contain a domain integration. This method therefore works provided u_p in (3) can be determined analytically, which is a significant task for a general function $f(\mathbf{x})$. In the next section, we address the issue of achieving a mesh free method for evaluating particular solutions. Once a particular solution has been determined, the homogeneous equations in (4)-(5) can be solved by standard BEMs. To avoid meshing surface in three-dimensions, we adopt the MFS, as demonstrated in Section 5.

3 Compactly Supported RBFs and the DRM

Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function such that $\varphi(0) \geq 0$. For any $\mathbf{x}_i \in \boldsymbol{\Omega}$, we let

$$\varphi_i(\mathbf{x}) = \varphi(\|\mathbf{x} - \mathbf{x}_i\|),$$

where $\|\bullet\|$ is the Euclidean norm; the functional φ_i being then the so-called radial basis function. Instead of defining a global function, CS-RBFs are radial basis functions with local support. In this section, we extend the results of [3] to a three-dimensional case in order to demonstrate the usefulness of CS-RBFs for large scale problems. For the detailed discussion of the CS-RBFs, we refer readers to References [13, 15, 14]. In Table 1, we give a list of CS-RBFs which were constructed by Wendland [14]. These functions contain the lowest possible degree among all piecewise polynomial CS-RBFs which are positively defined on $\mathbb{R}^d$ for a given order of smoothness.

Table 1. Wendland's CS-PD-RBFs; $r = \|\mathbf{x} - \mathbf{x}_i\|$.

$d = 1$	$\varphi = (1-r)_+$	C^0
	$\varphi = (1-r)_+^3\,(3r+1)$	C^2
	$\varphi = (1-r)_+^5\,(8r^2+5r+1)$	C^4
$d = 3$	$\varphi = (1-r)_+^2$	C^0
	$\varphi = (1-r)_+^4\,(4r+1)$	C^2
	$\varphi = (1-r)_+^6\,(35r^2+18r+3)$	C^4
	$\varphi = (1-r)_+^8\,(32r^3+25r^2+8r+1)$	C^6

In Table 1, we define

$$(1-r)_+^n = \begin{cases} (1-r)^n, & \text{if } 0 \le r \le 1, \\ 0, & \text{if } r > 1. \end{cases}$$

We also note that the radius of support in Table 1 has been normalized to [0,1]. In practical implementations, we need to rescale the support of φ with the support of a given radius α by using $\varphi(r/\alpha)$ for $\alpha > 0$. For a detailed discussion of this scaling effect, we refer the reader to [5, 13].

We now assume that $f(\mathbf{x})$ in (3) can be "approximated" by $\hat{f}(\mathbf{x})$ and that we can obtain an analytical solution $\hat{u}_p$ such that

$$\Delta \hat{u}_p(\mathbf{x}) = \hat{f}(\mathbf{x}). \tag{6}$$

The function u_p in (3) can then be approximated by $\hat{u}_p$ in (6) in the following sense. The initial step of the DRM is the determination of $f(\mathbf{x})$ through a basis function expansion. In three-dimeansions, a corresponding large number of interpolating points may be required in order that $f(\mathbf{x})$ may be obtained as accurately as possible. To avoid any associated ill-conditioning, we will employ CS-RBFs. The approximation of f by $\hat{f}$ is accomplished through the condition

$$f(\mathbf{x}_i) = \hat{f}(\mathbf{x}_i), \quad 1 \le i \le N, \tag{7}$$

where $\{\mathbf{x}_i\}_1^N$ is a given set of pairwise distinct centres The linear system

$$\hat{f}(\mathbf{x}_i) = \sum_{i=1}^{N} a_i \varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right), \quad 1 \le i \le N, \tag{8}$$

is then well-posed provided that the interpolation matrix

$$A_\varphi = \left[\varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)\right]_{1 \le i \le N} \tag{9}$$

is non-singular, which is the case when CS-RBFs are utilized (9). This then ensures the solvability of (9).

Once $\hat{f}$ in (6) has been determined, the function $\hat{u}_p$ can be computed by

$$\hat{u}_p = \sum_{i=1}^{N} a_i \Phi_i, \tag{10}$$

where

$$\Delta \Phi_i = \varphi_i \tag{11}$$

such that $\varphi_i = \varphi\left(\|\mathbf{x} - \mathbf{x}_i\|\right)$ and $\Phi_i = \Phi\left(\|\mathbf{x} - \mathbf{x}_i\|\right)$.

4 Evaluation of Particular Solutions

One of the key steps in the DRM is the analytical solution of (11); which we derive explicitly here. To be more specific, we choose $\varphi = (1-r)_+^4(4r+1)$ in Table 1, with scaling factor α. From (11), we have

$$\Delta\Phi\left(\frac{r}{\alpha}\right) = \begin{cases} \left(1-\dfrac{r}{\alpha}\right)^4\left(4\dfrac{r}{\alpha}+1\right), & r \leq \alpha, \\ 0, & r > \alpha. \end{cases} \tag{12}$$

Since then $\Delta = (1/r^2)(d/dr)(r^2 d/dr)$ in three-dimensions, by a straightforward integration we obtain

$$\begin{aligned} r^2\left(\frac{d}{dr}\Phi\left(\frac{r}{\alpha}\right)\right) &= \begin{cases} \displaystyle\int_0^r t^2\left(1-\frac{t}{\alpha}\right)^4\left(4\frac{t}{\alpha}+1\right)dt, & r \leq \alpha, \\ \displaystyle\int_0^\alpha t^2\left(1-\frac{t}{\alpha}\right)^4\left(4\frac{t}{\alpha}+1\right)dt + \int_\alpha^r 0dt, & r > \alpha. \end{cases} \\ &= \begin{cases} \dfrac{r^8}{2\alpha^5} - \dfrac{15r^7}{7\alpha^4} + \dfrac{10r^6}{3\alpha^3} - \dfrac{2r^5}{\alpha^2} + \dfrac{r^3}{3} & r \leq \alpha, \\ \dfrac{\alpha^3}{42}, & r > \alpha. \end{cases} \end{aligned} \tag{13}$$

Following the same integration procedure as above, we obtain

$$\begin{aligned} \Phi\left(\frac{r}{\alpha}\right) &= \begin{cases} \displaystyle\int_0^r \frac{1}{s^2}\left[\frac{s^8}{2\alpha^5} - \frac{15s^7}{7\alpha^4} + \frac{10s^6}{3\alpha^3} - \frac{2s^5}{\alpha^2} + \frac{s^3}{3}\right]ds, & r \leq \alpha, \\ \displaystyle\int_0^\alpha \frac{1}{s^2}\left[\frac{s^8}{2\alpha^5} - \frac{15s^7}{7\alpha^4} + \frac{10s^6}{3\alpha^3} - \frac{2s^5}{\alpha^2} + \frac{s^3}{3}\right]ds & \\ \displaystyle\quad + \int_\alpha^r \frac{1}{s^2}\left[\frac{\alpha^3}{42}\right]ds, & r > \alpha. \end{cases} \\ &= \begin{cases} \dfrac{r^2}{6} - \dfrac{r^4}{2\alpha^2} + \dfrac{2r^5}{3\alpha^3} - \dfrac{5r^6}{14\alpha^4} + \dfrac{r^7}{14\alpha^5}, & r \leq \alpha, \\ \dfrac{\alpha^2}{14} - \dfrac{\alpha^3}{42r}, & r > \alpha. \end{cases} \end{aligned} \tag{14}$$

5 The MFS for Laplace's Equation

In this section, we apply the MFS to solve the Laplace equation in (4)-(5). In the MFS, we embed the boundary of the domain into an auxiliary boundary $\partial\Omega_A \supset \Omega$. In general, we choose $\partial\Omega_A$ as a circle in two-dimensions [2, 7] and a sphere in three-dimensions. We then place "source" points on $\partial\Omega_A$. In general, these source points are evenly distributed on a sphere containing the domain Ω. The purpose of moving the source points outside of the domain Ω is to avoid the singularities of the fundamental solutions of the Laplacian.

Let $\{\mathbf{x}_j\}_{j=1}^m$ then denote a set of points on the auxiliary boundary $\partial\Omega_A$, so-called source points. We then approximate the solution $v(\mathbf{x})$ of (5) by a function of the form [4, 7]

$$v_m(\mathbf{x}) = \sum_{j=1}^{m} c_j G(\mathbf{x}, \mathbf{x}_j) + c, \quad \mathbf{x}_j \in \partial\Omega_A, \tag{15}$$

such that $G(\mathbf{x}, \mathbf{x}_j)$ is the fundamental solution;

$$G(\mathbf{x}, \mathbf{x}_j) = \begin{cases} \dfrac{1}{2\pi} \log \|\mathbf{x} - \mathbf{x}_j\|, & \mathbf{x}, \mathbf{x}_j \in \mathbb{R}^2, \\ \dfrac{-1}{4\pi\|\mathbf{x} - \mathbf{x}_j\|}, & \mathbf{x}, \mathbf{x}_j \in \mathbb{R}^3, \end{cases}$$

and $\|\bullet\|$ is the Euclidean norm. By collocation, we need to choose a set of points $\{\mathbf{x}_k\}_{k=1}^m$ on $\partial\Omega$. Applying the boundary conditions of (5) to (15), we obtain the following system of equations

$$\sum_{j=1}^{m} c_j G(\mathbf{x}_k, \mathbf{x}_j) + c = g(\mathbf{x}_k) - \widehat{u}_p(\mathbf{x}_k), \text{ for } k = 1, 2, ..., m. \tag{16}$$

Notice that due to the extra constant term c in (15), one more collocation point on the physical boundary is required. The above linear system of equations can be solved for $\{c_j\}_{j=1}^m \cup \{c\}$ by any linear solver. Bogomolny [2] showed that the auxiliary boundary $\partial\Omega_A$ can be taken as a circle in two-dimensions (or a sphere in three-dimensions) and the $\{\mathbf{x}_j\}_{j=1}^m$ equally distributed, and moreover, the larger the radius of the source circle (sphere), the better the approximation. In this case, however, the resulting matrix in (16) becomes extremely ill-conditioned. As indicated in [2], numerical result seems to be insensitive to this ill-conditioning.

An approximation u_m to u is then given by

$$u_m(\mathbf{x}) = \sum_{j=1}^{m} c_j G(\mathbf{x}, \mathbf{x}_j) + c + \hat{u}_p, \; \mathbf{x} \in \overline{\Omega}. \tag{17}$$

6 Numerical Example

In this section, we give an example to demonstrate the effectiveness of our proposed meshless method for a three-dimensional problem. It is known that the resulting matrix $\mathbf{A}_\varphi$ in (9) is sparse and iterative techniques such as the conjugate gradient method can be used to efficiently solve the system. We have implemented the following example on a PC with double precision. To solve the sparse system, we used a real sparse symmetric positive definite linear equation solver (DLSLDX) from the IMSL library (PC version).

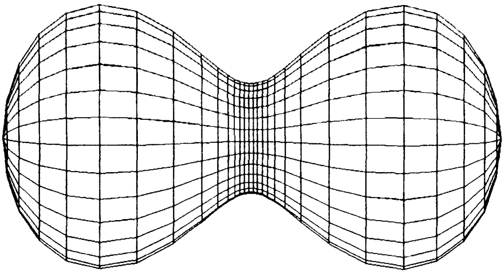

Figure 1: 3D graph of parametric surface in (20).

We consider the following problem:

$$\Delta u(x,y,z) = -3\cos(x)\cos(y)\cos(z), \quad (x,y,z)\in \mathbf{\Omega} \tag{18}$$
$$u(x,y,z) = \cos(x)\cos(y)\cos(z), \quad (x,y,z)\in \partial\mathbf{\Omega}. \tag{19}$$

Also, we denote $R(\theta) = \sqrt{\cos(2\theta)+\sqrt{1.1-\sin^2(2\theta)}}$. The surface of the domain we consider, $\mathbf{\Omega}\cup\partial\mathbf{\Omega}$, is represented by the following parametric surface

$$\mathbf{r}(\theta,\phi) = R(\theta)\cos(\theta)\ \mathbf{i} + R(\theta)\sin(\theta)\cos(\phi)\ \mathbf{j} + R(\theta)\sin(\theta)\cos(\phi)\ \mathbf{k}, \tag{20}$$

where $\theta\in[0,\pi), \phi\in[0,2\pi)$ (cf. Figure 1, 20). The analytical solution of (18)-(19) is given by

$$u(x,y,z) = \cos(x)\cos(y)\cos(z) \quad (x,y,z)\in \mathbf{\Omega}\cup\partial\mathbf{\Omega}. \tag{21}$$

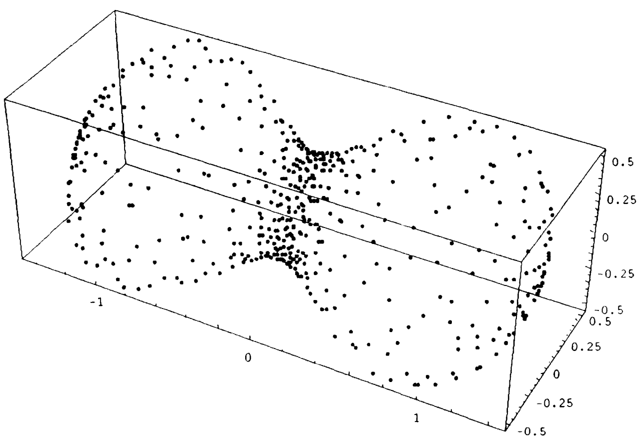

Figure 2: Quasi-random points on the surface as shown in (20).

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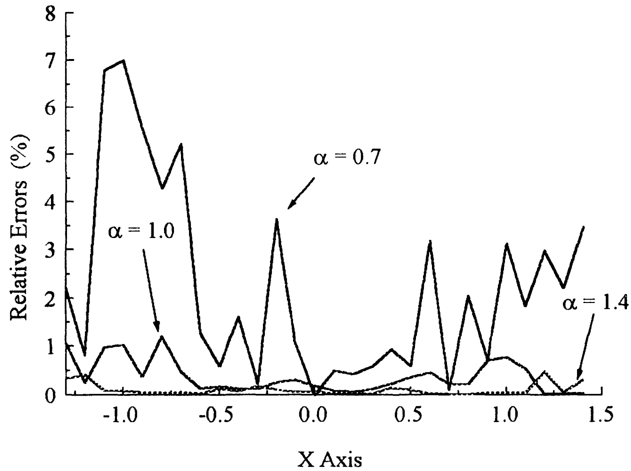

Figure 3: The effect of various scaling factor α.

In this example, we choose the basis function $\varphi = (1-r)_+^4(4r+1)$ applied to 300 quasi-random points [11] in the box $[-1.5, 1, 5] \times [-.5, .5] \times [.5, .5]$ for the interpolation of the forcing term. By collocation, particular solutions can be found using (10). In the MFS, we choose 101 quasi-random field points on the parametric surface, as shown in Figure 2, and 100 quasi-random source points on a sphere with center on origin and radius 9. The numerical results are computed along the x-axis with $y = z = 0$. The result of relative errors in percent with three different scaling factors are shown in Figure 3. We observe that with "larger" support, more interpolation points are included in the process of the approximation. Therefore, as expected, with more information a more accurate solutions is obtained.

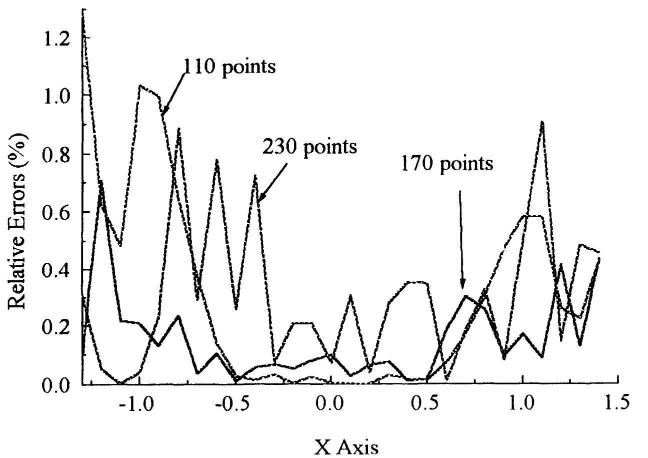

Figure 4: The effect of the number of source points using the MFS.

In Figure 4, we note in particular that greater accuracies are not achieved by increasing the number of collocation points in the MFS. As noted in [6], this is a consequence of the approximation error in the evaluation of the particular solution effecting the accuracy of u.

7 Conclusion

A truly meshless method has been proposed to solve Poisson's equation in three-dimesnions. Unlike methods which are mesh dependent, such as the finite element method, finite difference method, and the boundary element method, coding efficiencies result as a consequence of the elimination of the volume (or surface) discretization in the present approach. In addition, this method may be extended to solve other types of differential equations as shown in the DRM literature.

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