An extension of boundary element method for multiply connected regions

N. Kadioglu, S. Ataoglu

Faculty of Civil Engineering, Technical University of Istanbul, Turkey
Email:kadiog@itu.edu.tr
Email:ataoglu@itu.edu.tr

Abstract

An extension of boundary element formulation for linear elasticity problems is presented. The standard formulation for planar problems uses two kernels. First kernel is used to obtain the displacement components on the boundary and has logarithmic singularity. Introducing a new artificial boundary this singularity has been eliminated. Second kernel is used to determine unknown stress component on the boundary and has 1/r singularity. Defining a new coordinate system, this singularity has been also eliminated. Numerical results for a sample problem are presented.

Formulation

In this study, an improvement is introduced to solve the plane problems of linear elasticity by boundary element method.

It is considered that the problem to be solved is the first fundamental problem. In this kind of problems the external stress vector is given on the surface which represents the boundary of the region. It is chosen that region concerning the problem is multiply connected.

Now, the aim is to determine the displacement vector in the region and on the boundary. In addition to this, the components of the stress tensor must be determined in the region and also on the boundary.

Let $V$ be a region filled by a linear elastic material and $S$ be its boundary. In plane problems, $S$ and $V$ are a closed curve and a plane region respectively.
In the standard formulation of boundary element method any \( i \)th component of the displacement vector \( (i=1,2) \) at a point \( \tilde{y} \) in the region \( \bar{V} \) can be expressed as follows:

\[
\bar{u}_i(\tilde{y}) = \int_{\bar{S}} \frac{t^i(x, \tilde{y})}{\bar{r}^{i-1}} ds - \int_{\bar{S}} \int_{\bar{r}} (x, \bar{y}) \bar{u}(\bar{x}) ds
\]  

(1)

where, \( t(x) \) is the surface traction vector given on the boundary \( \bar{S} \). The kernels \( \bar{u}^i(x, \tilde{y}) \) and \( \bar{T}^i(x, \tilde{y}) \) are given below[1]

\[
\bar{u}^i_j(x, \tilde{y}) = -\frac{1}{8\pi\mu(1-\nu)} \left\{ (3-4\nu) \ln r \delta_{ij} - \frac{(x_i - y_i)(x_j - y_j)}{r^2} \right\}
\]  

(2)

\[
\bar{T}^i_{jk}(x, \tilde{y}) = -\frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu)(-\frac{(x_i - y_i)}{r^2} \delta_{jk} + \frac{(x_k - y_k)}{r^2} \delta_{ij}
\right. \\
\left. + \frac{(x_j - y_j)}{r^2} \delta_{ik}) + \frac{2(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^4} \right\} (i, j, k = 1,2)
\]  

(3)

\[
\bar{t}^i(x, \tilde{y}) = \bar{T}^i(x, \tilde{y}) n(x)
\]  

(4)

where \( \mu \) and \( \nu \) represent the shear modulus and Poisson's ratio, respectively. \( n(x) \) is the unit outward normal vector of the surface \( \bar{S} \) at the point \( \bar{x} \). \( \delta_{ij} \) represents Kronecker's delta.

Summation convention has been used in all expressions.

For a multiply connected region, the boundary \( \bar{S} \) contains a finite number of disjoint curves and the integral over \( \bar{S} \) is reduced to the summation of the integrals over this disjoint curves[2,3]. After examining the equation (1) it is clear that to obtain the displacement vector on the boundary it is enough to determine the displacement components at any point \( \tilde{y} \) of \( \bar{V} \). But before dealing with this problem, we have to explain the meanings of the kernels in the expression (1).

An infinite medium having the same constants \( \mu \) and \( \nu \) with the given problem is considered. \( \bar{x} \) and \( \bar{y} \) represent two different points of this medium.
$u^i(x, y)$ and $T^i(x, y)$ are the displacement vector and the stress tensor at point $x$ due to a singular body force of unit magnitude acting at point $y$ in the $i$ ($i = 1, 2$) direction, respectively.

For solving the integral equation (1), boundary $S$ is idealized as a collection of line segments. In this new boundary if the number of these line segments are $N$, the number of the end points of them will also be $N$. These end points are named as nodal points.

It is assumed that the variation of the displacement components on any of these linear segments is linear. Then the unknowns of the problem are reduced to the values of the displacement components on the nodal points.

$2N$ integral equations can be written by assuming there is a singular loading at every nodal point on each direction. In these integral equations, the integrals over the boundary are reduced to the summation of the integrals over the line segments.

In addition, we shall define a new artificial boundary which includes line segments but not the nodal points. Around each nodal point a small arbitrary curvilinear part which leaves the point outside is added to complete this new artificial boundary. It is assumed that displacement components are constant on this small curvilinear parts. After necessary calculations, these small curvilinear parts will be shrinken to the nodal points.

After calculating integrals over this artificial boundary we obtain a linear system of $2N$ equations with $2N$ unknowns which are the displacement components at nodal points.

After solving this linear system using the displacement field and the artificial boundary by the help of constitutive equations, one can calculate the stress components at any arbitrary internal point $y$ as follows:

$$T_{ij}(y) = \int_s \left[ t_k(x)u_k^i(x, y)ds - \int_s T_{kl}^i(x, y)n_l(x)u_k(x)ds \right]$$

(5)

The expressions of the kernels in eq. (5) are given below

$$u_k^i(x, y) = \frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu) \frac{(x_i-y_i)}{r^2} \delta_{jk} + \frac{(x_k-y_k)}{r^2} \delta_{ij} \right\} \frac{(x_j-y_j)}{r^2} \delta_{jk} - 2(x_i-y_i)(x_j-y_j)(x_k-y_k) \frac{1}{r^4}$$

(6)
If we want to calculate the stress components using the expressions (5) on the boundary we encounter $1/r$ singularity. To eliminate this singularity the stress components in a new $n, s$ coordinate system will be used instead of the stress components in $x_1, x_2$ (or $x$ and $y$) cartesian system. But this new coordinate system will be different for each linear segment (Figure 1).

![Figure 1. $n, s$ coordinate system near $J$ th line segment](image)

Let $n$ be the unit outward normal of any specific line segment whose number is $(J)$. $n$ axis coincides with $n$. $s$ is directed from nodal point $(J)$ towards nodal point $(J+1)$. The origin of $n, s$ system approaches to the nodal point $(J)$ for $\varepsilon = 0$. The stress component $T_{ss}(y)$ in $n, s$ system at any internal point $y$ can be calculated as follows:
\[ T_{ss}(y) = T_{11}(y)n_2^2(J) + T_{22}(y)n_1^2(J) - 2T_{12}(y)n_1(J)n_2(J) \]  \hspace{1cm} (8)

\[ T_{ss}(y) = \sum_{l=2}^{N+1} \int_{l(J)} \frac{u_k^{ss}(x,y)t_k(x)}{\partial s} + \sum_{l=2}^{N+1} \int_{l(J)} \frac{w_k^{ss}(x,y)}{\partial s} u_k(x,y) \]  \hspace{1cm} (9)

At a point on the boundary, we do not want to evaluate other stress components \( T^{n_s}(y) \) and \( T^{n_n}(y) \) which coincide with the \( n,s \) components of the surface traction vector \( t(x) \) which is known. The kernels of \( u_k^{ss}(x,y) \) and \( w_k^{ss}(x,y) \) have been given as follows:

\[ u_k^{ss}(x,y) = u_k^{11}(x,y)n_2^2(J) + u_k^{22}(x,y)n_1^2(J) - 2u_k^{12}(x,y)n_1(J)n_2(J) \]  \hspace{1cm} (10)

\[ w_k^{ss}(x,y) = w_k^{11}(x,y)n_2^2(J) + w_k^{22}(x,y)n_1^2(J) - 2w_k^{12}(x,y)n_1(J)n_2(J) \]  \hspace{1cm} (11)

\[ t_{ij}^{kl}(x,y) = T_{ij}^{kl}(x,y)n_j \]  \hspace{1cm} (12)

\[ t_{ij}^{kl}(x,y) = \frac{\partial}{\partial s}(w_{ij}^{kl}(x,y)) \]

\( n_1(J) \) and \( n_2(J) \) denote the components of \( n(J) \) which is the unit outward normal of a line segment whose number is \( J \). Now an interior point \( C \) is considered. The nearest point of the boundary to \( C \) be a point \( B \) on a line segment whose number is \( J \). \( B \) lies between \((J)th\) and \((J+1)th\) nodal points. But here we are introducing a restriction that the point \( B \) be neither \((J)th\) nor \((J+1)th\) nodal points. At this point \( C \), the stress component \( T_{ss}(y) \) will be calculated and \( \varepsilon \) which represents the shortest distance from \( C \) to the artificial boundary will be approached to zero in the limit.

After these calculations, stress component \( T_{ss}(y) \) is calculated without any singularity at a point of the \((J)th\) line segment of the artificial boundary. There is a distance \( s_0 \) between this specific point and the \((J)th\) nodal point. \( s_0 \) can be equal to neither zero nor \( l(J) \) which is the length of the \((J)th\) line segment.

All given formulas are valid for the plane strain. Replacing \( v \) by \( v/(1-v) \) will be enough to obtain the same expressions for plane stress.
Examples

Sample problem is a rectangular region having a rectangular hole under tension (Figure 2). Numerical values have been selected as $\mu=8000000$ N/cm$^2$ and $\nu=0.25$. The variations of various displacement components and stress components on some chosen lines have been given in figure (3), (4), (5), (6), (7), (8), (9), (10), (11) and (12).

![Figure 2. Rectangular region with a rectangular hole](image)

![Figure 3. Horizontal displacement on A-B and D-E lines](image)
Figure 4. Vertical displacement on A-B and D-E lines

Figure 5. Horizontal displacement on C-B and F-E lines

Figure 6. Vertical displacement on C-B and F-E lines
Figure 7. The variation of \( \frac{T_{11}}{\sigma_0} \) on some horizontal lines

Figure 8. The variation of \( \frac{T_{12}}{\sigma_0} \) on some horizontal lines
Figure 9. The variation of $T_{22}/\sigma_0$ on some horizontal lines

Figure 10. The variation of $T_{11}/\sigma_0$ on some vertical lines
Figure 11. The variation of $T_{12}/\sigma_0$ on some vertical lines

Figure 12. The variation of $T_{22}/\sigma_0$ on some vertical line

References

