On incremental equilibrium equations in solid mechanics

D. W. Nicholson
Mechanical, Materials and Aerospace Engineering
University of Central Florida, Orlando, FL. 32816
email: nicholsn@ucf.edu

Abstract

An incremental differential equilibrium equation is derived in solid mechanics for possible use in the boundary element method. The equation involves a tangent modulus tensor with two terms. One is totally symmetric, exhibiting at most twenty one distinct entries as in linear, anisotropic, inhomogeneous elasticity. There are only six distinct entries in the second term.

1. Introduction

The governing equation for a linear, anisotropic, inhomogeneous elastic medium under small strain involves a tangent modulus tensor with at most 21 distinct entries. The equation can potentially be solved using the boundary element method, following Divo and Kassab[1] and Kassab and Divo[2]. The goal of this investigation is to identify the extent to which the incremental equation in solid mechanics assumes the same form as the aforementioned equation. In consequence the present study bears on the possibility that the boundary element method can be extended to nonlinear problems treated by an incremental formulation. An important issue is the incremental tangent modulus tensor, for which a compact expression is derived using recent extensions of Kronecker Product Algebra[3]. The symmetry properties of the incremental tangent modulus tensor are identified. The results appear to be new.

Of course the issue of the proper form of the incremental equation has been addressed by many authors, for example Kleiber[4] and Hughes and Pister[5]. The goal in the earlier studies has often been to obtain correct incremental variational relations in integral form, which does not raise the question of the symmetry properties of the current tangent modulus tensor. Typically tensor-indicial notation has been used. We believe this type of notation hinders recognition of the structure of the resulting relations. The current study concerns the incremental differential equation corresponding to the Navier equation of linear elasticity. The symmetry properties of the tangent modulus tensor arising in the equation are addressed since they bear on the possibility of development of an incremental boundary element method. Finally, the equation and the associated tangent modulus tensor are obtained in compact form using extensions of Kronecker Product Algebra.

2. Navier's equation in an inhomogeneous, anisotropic linear elastic medium

First consider a linear, anisotropic, inhomogeneous elastic body under small strain. The linear strain tensor is denoted by ϵ and the associated stress by σ. The stress-strain relation is written as

$$\sigma = C_L(\mathbf{x})\epsilon \tag{1}$$

in which $\mathbf{C}_L(\mathbf{x})$ is a fourth order tensor. Now $\mathbf{C}_L$ has entries c_{ijkl} observing the following symmetry conditions:

$$c_{ijkl} = c_{jikl} \tag{2a}$$

$$c_{ijkl} = c_{ijlk} \tag{2b}$$

$$c_{ijkl} = c_{klij} \; . \tag{2c}$$

A fourth order tensor C satisfying (2a) will be called ***symmetric;*** it will be called ***totally symmetric*** if it satisfies (2a-2c). If C is totally symmetric with indices ranging from 1 to 3, it exhibits at most 21 distinct entries.

The equation of equilibrium and the strain-displacement relation are

$$(\partial/\partial\mathbf{x})\sigma^T = 0 \tag{3}$$

$$\epsilon = \frac{1}{2}[(\partial/\partial\mathbf{x})^T\boldsymbol{u}^T + ((\partial/\partial\mathbf{x})^T\boldsymbol{u}^T)^T] \ . \tag{4}$$

2.1 Incremental equations: basic relations

Invoking $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, Navier's equation follows after elementary manipulation as

$$(\partial/\partial\mathbf{x})\ \boldsymbol{C}_{\mathrm{L}}(\mathbf{x})\ (\partial/\partial\mathbf{x})^T\boldsymbol{u}^T = 0\,. \tag{5}$$

We now express the incremental equation of solid mechanics in a form similar to Eq. (5). Referring to the current configuration, let $\boldsymbol{t}$ denote the traction on a body whose surface element is denoted as dS. Static equilibrium requires that

$$\int \boldsymbol{t}dS = 0\ . \tag{6}$$

We recall that $\boldsymbol{t} = \boldsymbol{\tau}^T\boldsymbol{n}$ in which $\boldsymbol{\tau}$ is the (symmetric) Cauchy stress and $\boldsymbol{n}$ is the exterior surface normal vector. Now invoking Nanson's theorem[6] $\boldsymbol{n}\mathrm{d}S = J\boldsymbol{F}^{-T}\boldsymbol{n}_0 dS_0$, we introduce the nonsymmetric 1st Piola Kirchhoff stress $\boldsymbol{\sigma}$ using

$$\boldsymbol{\tau}^T\boldsymbol{n}dS = \boldsymbol{\sigma}^T\boldsymbol{n}_0 dS_0. \tag{7}$$

Now $\boldsymbol{\sigma} = J\boldsymbol{F}^{-1}\boldsymbol{\tau}$, and $\boldsymbol{n}_0$ is the exterior normal vector to the surface element dS_0 in the undeformed configuration. Also, $\boldsymbol{F}$ is the deformation gradient tensor with $J = det(\boldsymbol{F})$. The 2nd Piola-Kirchhoff stress tensor σ^*, which is conjugate to the Lagrangian strain tensor, is given by $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}\boldsymbol{F}^{-T}$. Static equilibrium now implies that

$$\int \boldsymbol{\sigma}^T\boldsymbol{n}_0 dS_0 = 0\,. \tag{8}$$

We now invoke the notion of an increment following Kleiber[5]. Suppose the solution for $\boldsymbol{\sigma}$ is known as $\boldsymbol{\sigma}_{\mathrm{n}}$ at time t_n and is sought at t_{n+1}. We introduce the increment $\Delta\boldsymbol{\sigma}$ to denote $\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$. Now equilibrium applied to $\boldsymbol{\sigma}_{n+1}$ and $\boldsymbol{\sigma}_n$ implies

Transactions on Modelling and Simulation vol 20, © 1998 WIT Press, www.witpress.com, ISSN 1743-355X

$$\int \Delta\sigma^T \boldsymbol{n}_0 dS_0 \; = \; 0\,. \tag{9}$$

Recalling Eq. (3), application of the Divergence Theorem furnishes the differential equation

$$(\partial/\partial\mathbf{x})\Delta\sigma^T \; = \; 0 \tag{10}$$

in which $\mathbf{x}$ is now understood to denote the *undeformed* coordinates. Finally, we require increments of the deformation gradient tensor $\boldsymbol{F}$ and the Lagrangian strain ϵ. In terms of the increment $\Delta\mathbf{u}$ and neglecting quadratic terms in increments,

$$\begin{aligned} \Delta\boldsymbol{F}^T &= \; (\partial/\,\partial\mathbf{x})^T \Delta\mathbf{u}^T \\ \Delta\epsilon &= \frac{1}{2}[\,(\Delta\boldsymbol{F})^T\boldsymbol{F} + \boldsymbol{F}^T\Delta\boldsymbol{F} + (\Delta\boldsymbol{F})^T\Delta\boldsymbol{F}] \\ &\simeq \tfrac{1}{2}[\,(\Delta\boldsymbol{F})^T\boldsymbol{F} + \boldsymbol{F}^T\Delta\boldsymbol{F}\,]\,. \end{aligned} \tag{11}$$

The next task is to determine an incremental tangent modulus tensor $\boldsymbol{C}(\boldsymbol{x})$ such that

$$\Delta\sigma^T = \; \boldsymbol{C}(\boldsymbol{x})(\Delta\boldsymbol{F})^T \;\; . \tag{12}$$

It is immediately evident that $\boldsymbol{C}(\boldsymbol{x})$ *cannot be totally symmetric since* $\Delta\sigma$ *is not symmetric.* Equation (12) contrasts with, and is more formidable than, the problem of incremental variational principles[4,5]. In the latter case, the increment of the *symmetric* 2nd Piola-Kirchhoff stress tensor is required as a function of the increment of the *symmetric* Lagrangian strain tensor.

Recalling Eq. (5), we now obtain

$$(\partial/\partial\mathbf{x})\; \boldsymbol{C}(\mathbf{x})\; (\partial/\partial\mathbf{x})^T(\Delta\boldsymbol{u})^T = \;\; 0\,. \tag{13}$$

Transactions on Modelling and Simulation vol 20, © 1998 WIT Press, www.witpress.com, ISSN 1743-355X

3. Truesdell stress rate

There are many different possibilities for deriving $C(x)$, depending on how the strain-strain relations are regarded as measured. Here we assume that a rate relation is measured in the current configuration. However, owing to the requirement of objectivity it is not appropriate to express the time rate of the Cauchy stress tensor $\partial\boldsymbol{\tau}/\partial t$ directly in terms of the deformation rate tensor $\boldsymbol{D}$. Instead, it is necessary to introduce an objective stress rate, for which there are many possibilities. Here, for reasons addressed in Nicholson and Lin[7] we invoke the *Truesdell stress flux* given by

$$\partial\boldsymbol{\theta}/\partial t \;=\; \partial\boldsymbol{\tau}/\partial t \;+\; \boldsymbol{\tau}\mathrm{tr}(\boldsymbol{D}) \;-\; \boldsymbol{L}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{L}^T \tag{14}$$

in which $\boldsymbol{L}$ is the velocity gradient tensor and $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$. We now assume that a fourth order tangent modulus tensor $\boldsymbol{C}_0(\boldsymbol{x})$ is known such that

$$\partial\boldsymbol{\theta}/\partial t \;=\; \boldsymbol{C}_0(\boldsymbol{x})\boldsymbol{D} \tag{15}$$

We assume that $\boldsymbol{C}_0(\boldsymbol{x})$ is totally symmetric and hence involves at most 21 distinct entries (which depend on $\mathbf{x}$, at least implicitly). Conversion to undeformed coordinates is facilitated by using the transformations[7].

$$\partial\boldsymbol{\sigma}^*/\partial t \;=\; J\boldsymbol{F}^{-1}\partial\boldsymbol{\theta}/\partial t\,\boldsymbol{F}^{-T} \qquad \partial\epsilon/\partial t \;=\; \boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}\,. \tag{16}$$

We now introduce the obvious approximations

$$\Delta\boldsymbol{\sigma}^* \;\simeq (\partial\boldsymbol{\sigma}^*/\partial t)\Delta t \qquad\qquad \Delta\epsilon \;\simeq (\partial\epsilon/\partial t)\Delta t\,. \tag{17}$$

in which $\Delta t = t_{n+1} - t_n$. Consequently,

$$\begin{aligned} \boldsymbol{F}\Delta\boldsymbol{\sigma}^*\boldsymbol{F}^{\mathrm{T}}/J \;&=\; \boldsymbol{C}_0[\boldsymbol{F}^{-T}\Delta\epsilon\boldsymbol{F}^{-1}] \\ &=\; \boldsymbol{C}_0\boldsymbol{F}^{-T}\Delta\boldsymbol{F}^T\,. \end{aligned} \tag{18}$$

Finally,

$$\Delta\sigma^* = JF^{-1}(C_0F^{-T}\Delta F^{T)}F^{-T}. \tag{19}$$

Again neglecting quadratic products in increments, the 1st and 2nd Piola-Kirchhoff stress tensors satisfy

$$\Delta\sigma^* \simeq \Delta\sigma F^{-T} + \sigma\Delta F^{-T}. \tag{20}$$

Since $\Delta(F^TF^{-T}) = 0$,

$$\Delta F^{-T} \simeq -F^{-T}\Delta F^TF^{-T}. \tag{21}$$

Exploiting the total symmetry of C_0, we obtain

$$\Delta\sigma = JF^{-1}C_0F^{-T}\Delta F^T + \sigma F^{-T}\Delta F^T . \tag{22}$$

4. Kronecker Product Algebra

We seek to use Eq. (22) to derive a compact expression for $C(x)$ knowing $C_0(x)$. For this purpose it is convenient to invoke extensions of Kronecker product algebra, introduced recently in Nicholson and Lin[3]. The operator *VEC(.)* converts a second order $n \times n$ tensor A into a vector ($n \times 1$ tensor) a as follows.

$$a = VEC(A) = \{a_{11}\ a_{21}\ a_{31}\ .....a_{nn}\}^T . \tag{23}$$

In addition, there exists a universal $n^2 \times n^2$ permutation tensor U such that $VEC(A^T) = UVEC(A)$. Now U has the remarkable property that $U^T = U^{-1} = U$ and hence $U^2 = I$.

The Kronecker product $\otimes$ of two second order $n \times n$ tensors A and B generates a second order $n^2 \times n^2$ tensor given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B. & . & . & . \\ a_{21}B. & . & . & . & . \\ . & . & . & . & . \\ . & . & & & . \\ . & . & . & . & a_{nn}B \end{bmatrix} \tag{24}$$

A fourth order n × n × n × n tensor $\boldsymbol{H}$ gives rise to an $n^2 \times n^2$ tensor Γ by virtue of the *TEN22* operator[3] defined implicitly as follows. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are second order n × n tensors satisfying $\boldsymbol{A} = \boldsymbol{HB}$, then $\Gamma = TEN22(\boldsymbol{H})$ satisfies $\boldsymbol{a} = \Gamma \boldsymbol{b}$ where $\boldsymbol{a} = VEC(\boldsymbol{A})$ and $\boldsymbol{b} = VEC(\boldsymbol{B})$. Conversely, the inverse operator *ITEN22(.)* satisfies $\boldsymbol{H} = ITEN22(\Gamma)$ if $\Gamma = TEN22(\boldsymbol{H})$.

Several relations are quoted for use in subsequent paragraphs[3]. Here $\boldsymbol{I}$ is the second order n × n identity tensor.

$$\boldsymbol{UA} \otimes \boldsymbol{BU} = \boldsymbol{B} \otimes \boldsymbol{A} \tag{25a}$$

$$(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1} \tag{25b}$$

$$TEN22(\boldsymbol{C}^{-1}) = TEN22^{-1}(\boldsymbol{C}) \tag{25c}$$

$$TEN22(\boldsymbol{ACB}) = \boldsymbol{I} \otimes \boldsymbol{A}\, TEN22(\boldsymbol{C})\boldsymbol{I} \otimes \boldsymbol{B}\,. \tag{25d}$$

Also, suppose $\boldsymbol{C}$ has entries c_{ijkl} and δ_{ij} is the ijth entry of the Kronecker (substitution) tensor. If $c_{ijkl} = \delta_{ik}\sigma^*_{jl}$, then

$$TEN22(\boldsymbol{C}) = \boldsymbol{\sigma}^* \otimes \mathbf{I}\,. \tag{25e}$$

We now introduce a new result of interest in the present context.. The fourth order tensor $\boldsymbol{C}$ is totally symmetric if and only if

$$TEN22(\boldsymbol{C}) = TEN22^T(\boldsymbol{C}) \tag{26a}$$

$$\boldsymbol{U}TEN22(\boldsymbol{C}) = TEN22(\boldsymbol{C}) \tag{26b}$$

$$TEN22(\boldsymbol{C})\boldsymbol{U} = TEN22(\mathbf{C}). \tag{26c}$$

Equation (26a) is equivalent to symmetry with respect to exhange of ij and kl in $\boldsymbol{C}$. Total symmetry also implies that, for any second order n × n tensor $\boldsymbol{B}$, the corresponding tensor $\boldsymbol{A} = \boldsymbol{CB}$ is symmetric. Thus, if $\boldsymbol{a} = VEC(\boldsymbol{A})$ and $\boldsymbol{b} = VEC(\boldsymbol{B})$, then $\boldsymbol{a} = TEN22(\boldsymbol{C})\boldsymbol{b}$ and $\boldsymbol{Ua} = TEN22(\boldsymbol{C})\boldsymbol{b}$. Multiplying through the latter expression with $\boldsymbol{U}$ implies (26b). Next, for any n × n tensor $\boldsymbol{A}$, the tensor $\boldsymbol{B} = \boldsymbol{C}^{-1}\boldsymbol{A}$ is symmetric. It follows that $\boldsymbol{b} = TEN22(\mathbf{C}^{-1})\boldsymbol{a} = TEN22^{-1}(\boldsymbol{C})\boldsymbol{a}$, and $\boldsymbol{Ub} = TEN22^{-1}(\boldsymbol{C})\boldsymbol{a}$. Thus $TEN22^{-1}(\boldsymbol{C}) = \boldsymbol{U}TEN22^{-1}(\boldsymbol{C})$. It follows that $TEN22(\boldsymbol{C}) = [\boldsymbol{U}TEN22^{-1}(\boldsymbol{C})]^{-1} = TEN22(\boldsymbol{C})\boldsymbol{U}$. We now draw the immediate conclusion from (2b) and (2c) that $\boldsymbol{U}TEN22(\boldsymbol{C})\boldsymbol{U} = TEN22(\boldsymbol{C})$ if $\boldsymbol{C}$ is totally symmetric.

We next prove the following:

$$\boldsymbol{C}^{-1} \text{ is totally symmetric if } \boldsymbol{C} \text{ is totally symmetric.} \tag{27}$$

As proof note that $TEN22(\boldsymbol{C})\boldsymbol{U} = TEN22(\boldsymbol{C})$ implies that $\boldsymbol{U}TEN22(\boldsymbol{C}^{-1}) = TEN22(\boldsymbol{C}^{-1})$, while $\boldsymbol{U}TEN22(\boldsymbol{C}) = TEN22(\boldsymbol{C})$ implies that $TEN22(\boldsymbol{C}^{-1})\boldsymbol{U} = TEN22(\boldsymbol{C}^{-1})$.

Finally, we prove the following : for nonsingular n × n tensor $\boldsymbol{G}$,

$$\boldsymbol{GCG}^T \text{ is totally symmetric if } \boldsymbol{C} \text{ is totally symmetric.} \tag{28}$$

First, Eq. (25d) implies that $TEN22(\boldsymbol{GCG}^{-T}) = \boldsymbol{I}\otimes\boldsymbol{G}TEN22(\boldsymbol{C})\boldsymbol{I}\otimes\boldsymbol{G}^T$, so that $TEN22(\boldsymbol{GCG}^T)$ is certainly symmetric. Next consider whether $\boldsymbol{A}'$ given

$$\boldsymbol{A}' = \boldsymbol{GCG}^T\boldsymbol{B}' \tag{29}$$

is symmetric in which $\boldsymbol{B}'$ is a second order nonsingular n × n tensor. But we may write

$$\boldsymbol{G}^{-1}\boldsymbol{A}'\boldsymbol{G}^{-T} = \boldsymbol{C}\,\boldsymbol{G}^T\boldsymbol{B}'\boldsymbol{G}^{-T}. \tag{30}$$

Now $\boldsymbol{G}^{-1}\boldsymbol{A}'\boldsymbol{G}^{-T}$ is symmetric since $\boldsymbol{C}$ is totally symmetric, and therefore $\boldsymbol{A}'$ is symmetric. Next consider whether $\boldsymbol{B}'$ given by the following is symmetric.

$$\boldsymbol{B}' = \boldsymbol{G}^{-T}\boldsymbol{C}^{-1}\boldsymbol{G}^{-1}\boldsymbol{A}'. \tag{31}$$

But we may write

$$\boldsymbol{G}^T\boldsymbol{B}'\boldsymbol{G} = \boldsymbol{C}^{-1}\boldsymbol{G}^{-1}\boldsymbol{A}'. \tag{32}$$

Since $\boldsymbol{C}^{-1}$ is totally symmetric, it follows that $\boldsymbol{G}^T\boldsymbol{B}'\boldsymbol{G}$ is symmetric, and hence $\boldsymbol{B}'$ is symmetric. We conclude that $\boldsymbol{GCG}^T$ is totally symmetric.

5. Incremental equilibrium equation

Now recalling Eqs. (22,25a), letting $\boldsymbol{s} = VEC(\boldsymbol{\sigma})$ and $\boldsymbol{f} = VEC(\boldsymbol{F})$ we obtain

$$\Delta\boldsymbol{s} = \boldsymbol{\Gamma}'\Delta\boldsymbol{f}$$

$$\boldsymbol{\Gamma}' = [\boldsymbol{I}\otimes\boldsymbol{\sigma}^* + J\boldsymbol{I}\otimes\boldsymbol{F}^{-1}TEN22(\boldsymbol{C}_0)\boldsymbol{I}\otimes\boldsymbol{F}^{-T}]\boldsymbol{U}. \tag{33}$$

$$= [\boldsymbol{I}\otimes\boldsymbol{\sigma}^* + J\,TEN22(\boldsymbol{F}^{-1}\boldsymbol{C}_0\boldsymbol{F}^{-T})]\boldsymbol{U}. \tag{34}$$

The matrix $\boldsymbol{\Gamma}'$ is not symmetric. However, note that $\boldsymbol{\Gamma} = U\boldsymbol{\Gamma}'$ is symmetric. Further, note that in Eqn (12) $\Delta\sigma^T$ appears, and that $VEC(\Delta\sigma^T) = UVEC(\Delta\sigma)$. Thus the matrix which actually appears in the incremental equilibrium equation is $\boldsymbol{\Gamma}$, which is symmetric. We next discuss how many distinct entries $\boldsymbol{C}(\mathbf{x}) = ITEN22(\boldsymbol{\Gamma})$ possesses.

Referring to Eq. (12) we thus conclude that

$$\begin{aligned} C(x) &= ITEN22(U[\boldsymbol{I} \otimes \sigma^* + JTEN22(\boldsymbol{F}^{-1}C_0\boldsymbol{F}^{-T})]\boldsymbol{U}) \\ &= ITEN22(\boldsymbol{\sigma}^* \otimes \boldsymbol{I}) + J\boldsymbol{F}^{-1}C_0\boldsymbol{F}^{-T}. \end{aligned} \tag{35}$$

Equation (28) implies that $\boldsymbol{F}^{-1}C_0\boldsymbol{F}^{-T}$ is totally symmetric and hence contains at most 21 distinct entries. Unfortunately, $ITEN22(\boldsymbol{\sigma}^* \otimes \boldsymbol{I})$ is only symmetric! Note from Eq. (25c) that the ijkl entry of $ITEN22(\boldsymbol{\sigma}^* \otimes \boldsymbol{I})$ is $\delta_{ik}\sigma^*_{jl}$.

6. Conclusion

An incremental differential equilibrium equation has been derived in solid mechanics. The equation involves a tangent modulus tensor with two terms. One is totally symmetric, exhibiting at most twenty one distinct entries. The second term has six distinct entries.

7. References

1. Divo, E. and Kassab, A., "A Boundary Integral Equation for Steady State Heat Conduction in Anisotropic and Inhom- ogeneous Media", Numer Meth Heat Trans, Vol 32, No 1, pp. 37-61, 1997.

2. Kassab ,A. and Divo, E., "A General Boundary Integral Equation for Isotropic Heat Conductiion in Isotropic Heat Conduction Problems in Bodies with Space Dependent Properties", Engineering Analysis, Vol 18, No 4, pp. 273-286, 1996.

Transactions on Modelling and Simulation vol 20, © 1998 WIT Press, www.witpress.com, ISSN 1743-355X

3. Nicholson, D.W. and Lin, B., "Extensions of Kronecker Product Algebra, with Applications in Contimuum and Computational Mechanics", in review.

4. Kleiber, M, *Incremental Finite Element Modelling in Nonl-Linear Solid Mechanics,* Ellis Horwood Ltd, John Wiley and Sons, New York, 1898.

5. Hughes, T.J., and Pister, K.S., "Consistent Linearization in Mechanics of Solids and Structures", Comp Struct, Vol 8, 391-397, 1978.

6. Chandrasekharaiah, D. and Debnath, L., *Continuum Mechanics,* Academic Press, Liimited, London, 1994.

7. Nicholson, D.W. and Lin, B., "Tangent Modulus Tensor in Plasticity under Finite Strain", to appear in Acta Mechanica.