Solution of 2D, 3D and axisymmetric finite deformation problems by the boundary-domain integral method

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Abstract

A boundary-domain integral method for 2D, 3D and axisymmetric problems at finite strains is presented, which is applicable to arbitrary hyperelastic constitutive laws. By introducing an intermediate configuration, elastoplastic problems can be treated, too. A total Lagrange scheme is used, so that the arising integrals have to be evaluated only once, which minimizes the numerical effort to compute the system matrices. In an incremental solution procedure the nonlinear set of equations is solved by Newton’s method in each load step. A significant reduction of computation time could be achieved by using an iterative solver (GMRES) within Newton’s method to compute the corrections from the linear equation with the Jacobian as matrix of coefficients. The current implementation also includes nonlinear boundary conditions (e.g. follower loads) and special solution techniques for instability problems (e.g. snap-through of structures) like the arc-length method. The numerical examples comprise finite deformation problems with elastoplastic as well as incompressible rubberlike materials. The results obtained by the boundary-domain integral method will be compared to those obtained by the finite element method.

1 Governing equations

The presented boundary-domain integral method for geometrically and physically nonlinear problems is based on a total Lagrangian
Boundary Elements

approach, i.e. the basic equations refer to the undeformed configuration of the body. For 2D and 3D problems a fixed cartesian frame is used to describe the deformation. In the initial configuration, $X_i$ denote the coordinates of a material particle $X$ of the undeformed body. After the deformation, this particle occupies the new position $x_i$. The equation of equilibrium $\partial \sigma_{ij}/\partial x_i + b_j = 0$ expressed in terms of Cauchy stresses $\sigma_{ij}$ in the deformed configuration can be transformed to the initial configuration by making use of the deformation gradient

$$F_{ij}(X) = \frac{\partial x_i(X)}{\partial X_j} , \quad (1)$$

resulting in

$$(S_{ik} F_{jk})_i + b^0_j = 0 , \quad (2)$$

where $b^0_j$ denotes the body force with respect to the undeformed body and $S_{ij}$ is the symmetric 2nd Piola–Kirchhoff stress tensor

$$S_{ij} = J (F^{-1})_{ik} \sigma_{kl} (F^{-1})_{jl} \quad \text{with} \quad J = \det(F_{ij}) , \quad (3)$$

which is defined in the undeformed configuration. Since the equations are based on the undeformed configuration, partial derivatives denoted by $(...)_j$ are taken with respect to the undeformed coordinates $X_j$. Using the displacement $u_i$ of a material particle $X$, such that $x_i = X_i + u_i$, leads to the alternative expression for the deformation gradient

$$F_{ij}(X) = \delta_{ij} + \frac{\partial u_i(X)}{\partial X_j} = \delta_{ij} + u_{i,j}(X) . \quad (4)$$

The relation between the tractions $t^0_i$, which measure the force per unit undeformed area, the tractions $t_i$ in the deformed configuration and the stress tensors $\sigma_{ki}$ and $S_{kj}$ is

$$t^0_i = F_{ij} S_{kj} N_k = t_i \frac{da}{dA} = \sigma_{ki} n_k \frac{da}{dA} . \quad (5)$$

$N_k$ are the components of the unit vector normal to the corresponding undeformed area element $dA$ in the initial configuration, and $n_k$ are the components of the normal vector to $dA$ in the deformed configuration.

Finally, a general hyperelastic constitutive law is used to determine the 2nd Piola–Kirchhoff stress tensor

$$S_{ij} = 2\rho \frac{\partial W(X, C)}{\partial C_{ij}} , \quad (6)$$
where the stored energy function $W$ depends on the right Cauchy-Green tensor $C_{ij} = F_{ki}F_{kj}$. For incompressible materials like e.g. rubber the hyperelastic constitutive equation reads

$$S_{ij} = 2\rho \frac{\partial W(X, C)}{\partial C_{ij}} - p(C^{-1})_{ij},$$

(7)

where the pressure $p$ is an unknown function, that must be determined by the condition of incompressibility. In order to treat elastoplastic problems, an intermediate configuration is introduced by the local multiplicative decomposition $F_{ij} = F_{ki}^e F_{kj}^p$ of the deformation gradient into an elastic and a plastic part. Since this decomposition is not unique, the plastic right Cauchy-Green tensor $C_{ij}^p = F_{ki}^p F_{kj}^p$ is used instead to describe the intermediate configuration. The stored energy function $W$ in the hyperelastic part of the elastoplastic constitutive equations in this case also depends on the plastic right Cauchy-Green tensor $C_{ij}^p$ and on internal variables $A$, which are determined by a set of evolution equations $\dot{C}^p = \mathcal{F}(C, C^p, A, \dot{C})$ and $\dot{A} = \mathcal{G}(C, C^p, A, \dot{C})$ (see e.g. Simo [1, 2]). The algorithmic integration of the evolution equations and the computation of the consistent material tangent moduli depend on the explicit structure of the evolution equations.

The derivation of the boundary-domain integral equations requires the constitutive laws (6) or (7), respectively, to be split into a linear and a nonlinear part

$$S_{ij} = C_{ijkl} \frac{1}{2}(u_{k,l} + u_{l,k}) + S_{ij}^n \quad \text{with} \quad C_{ijkl} = 2 \frac{\partial S_{ij}}{\partial C_{kl}} \bigg|_{C=C^p=l} ,$$

(8)

where the nonlinear stress tensor

$$S_{ij}^n = S_{ij} - C_{ijkl} \frac{1}{2}(u_{k,l} + u_{l,k})$$

(9)

is suitably chosen, to ensure the equivalence of eqn (8) and the hyperelastic constitutive equation. The elasticity tensor $C_{ijkl}$ in eqn (8) is assumed to be isotropic and homogeneous.

2 Boundary-domain integral equations

The boundary-domain integral equation for finite deformation problems can be derived from the weak formulation of the balance of linear
momentum (2) 
\[ \int_{\Omega} \left[ (S_{kl} F_{jl})_{,k} + b^0_j \right] U_j \, d\Omega = 0 , \] (10)

using as weight function \( U_j \) Kelvin’s fundamental solution to Lamé–Navier’s equation of linear elasticity with the constant elasticity tensor \( C_{ijkl} \) from eqn (8). Due to the total Lagrangian approach, \( \Omega \) is the domain of the undeformed body, and all derivatives are taken with respect to the undeformed configuration. For the sake of simplicity, the body force \( b^0_j \) will be omitted in the following. The derivation of the boundary-domain integral equation can be performed in a classical way for the linear part of the governing equations, which finally leads to

\[ c_{ij}(\xi) u_j(\xi) = \int_{\Gamma} U_{i,j} \, e^0_j(X) \, d\Gamma - \int_{\Gamma} T_{ij} \, u_j(X) \, d\Gamma - \int_{\Omega} U_{ij,k} \, N_l_{jk}(X) \, d\Omega , \] (11)

where \( i \) denotes the direction of the unity load in the linear reference problem. The nonlinearity leads to the additional domain integral with the nonlinear term \( N_l_{jk} \)

\[ N_l_{jk}(X) = \left[ \delta_{jl} + u_{j,l}(X) \right] S_{kl}(X) - C_{k,ijm} u_{l,m}(X) . \] (12)

As the nonlinear term \( N_l_{jk} \) in the domain integral of eqn (11) depends on the derivatives of the displacement, a representation formula for the displacement gradient is required. For interior source points, the generalized Somigliana identity can be differentiated with respect to the source point coordinates, which leads to the boundary-domain integral equation for displacement derivatives

\[ \frac{\partial u_i(\xi)}{\partial \xi_l} = \int_{\Gamma} \left[ \frac{\partial U_{i,j}}{\partial \xi_l} \, e^0_j(X) - \frac{\partial T_{ij}}{\partial \xi_l} \, u_j(X) \right] \, d\Gamma - \int_{\Omega_R} \frac{\partial U_{ij,k}}{\partial \xi_l} \, N_l_{jk}(X) \, d\Omega_R \]
\[ - \int_{\Omega_S} \frac{\partial U_{ij,k}}{\partial \xi_l} \left[ N_l_{jk}(X) - N_l_{jk}(\xi) \right] \, d\Omega_S - \int_{\Gamma_S} \frac{\partial U_{ij}}{\partial \xi_l} \, N_k(X) \, d\Gamma_S , \] (13)

where the domain has been split into a regular part \( \Omega_R \) and a singular part \( \Omega_S \), which contains the source point \( \xi \). \( N_k \) is the normal vector of the boundary \( \Gamma_S \) of the singular domain \( \Omega_S \). In eqn (13), the strongly singular domain integral over \( \Omega_S \) has been regularized via Gauss theorem [3, 4].
If the appropriate kernel functions are used, axisymmetric problems can be treated, too. Only the regularization of the strongly singular domain integral requires special care, when applying Gauss theorem. Three different expressions are obtained for the regularized domain integral depending on the index $k = r, \varphi$ and $z$ of the axisymmetric kernel $\bar{U}_{ijk}^\alpha$ [4]. In contrast to small deformation problems, an additional term appears in the equation of equilibrium (2) and therefore also in the nonlinear term $N_{ijk}$ of the domain integral, when an axisymmetric problem with twist is considered. The reason for this is, that radial planes of the body with normal vectors $N_{\varphi}$ in the initial configuration are distorted during the deformation. This influences the calculation of derivatives in curvilinear coordinate systems, resulting in the additional terms [5].

3 Discretization and numerical solution

For the discretization of eqns (11) and (13) for 2D/3D problems or the related axisymmetric equations, respectively, the standard procedures are used. Applying the collocation technique to the discretized integral equations provides the matrix equations

$$A\mathbf{u} = B\mathbf{t} - E\mathbf{n} \quad \Rightarrow \quad \tilde{A}\mathbf{x} = \tilde{B}\mathbf{y} - E\mathbf{n}, \quad (14)$$

and

$$\mathbf{h} = \tilde{G}\mathbf{y} + \tilde{H}\mathbf{x} - E\mathbf{n}. \quad (15)$$

For the nodes of the boundary elements, the vectors $\mathbf{u}$, $\mathbf{t}$, $\mathbf{y}$ and $\mathbf{x}$ contain the values of the boundary displacements and tractions and the known and unknown boundary values, respectively. The vectors $\mathbf{h}$ and $\mathbf{n}$ contain the values of the displacement gradients and the nonlinear terms for the nodes of the internal cells. The evaluation of the hypersingular boundary-domain integral equation for displacement gradients on the boundary can be avoided by using nonconforming cells. Eliminating the unknown boundary values $\mathbf{x}$ in eqn (15) by using eqn (14) leads to a nonlinear set of equations

$$0 = \mathbf{f} = \mathbf{h} + (i)E\mathbf{n} - \mathbf{h}(\lambda), \quad (16)$$

with a residual vector $\mathbf{f}$ and the abbreviations

$$\tilde{E} = \tilde{H}\tilde{A}^{-1}\mathbf{E} + \mathbf{E}'; \quad (i)\mathbf{h}(\lambda) = \left[\tilde{G} + \tilde{H}\tilde{A}^{-1}\tilde{B}\right] (i)\mathbf{y}(\lambda). \quad (17)$$
The matrix $\tilde{E}$ is constant, and $h^{(l)}(\lambda)$ depending on the load factor $\lambda$ can be calculated directly from the prescribed boundary values once the system matrices have been computed.

The path dependence of inelastic problems requires an incremental solution strategy for eqn (16), taking into account the integration algorithm of the evolution equation. In the typical $(k + 1)^{th}$ load increment the vector of basic unknowns $h^{k+1}$ has to be determined as the solution of

$$f^{k+1} = h^{k+1} + \tilde{E}n(h^{k+1}, s(c^{k+1}, c^p, \Lambda^{k+1})) - h^{(l)} = 0 , \quad (18)$$

in which $c^{k+1}$ is a function of $h^{k+1}$. The stresses $s$ depend on $c^{k+1}$ and on the intermediate configuration characterized by $c^b, \Lambda^{k+1}$, which result from a local time integration algorithm of the evolution equations at each internal node. The integration starts from the known nodal values $(C, C^p, \Lambda)^k$ at the end of the $k^{th}$ load increment to avoid influences of the iteration path. The vector $h^{(l)k+1}$ represents the new loading in the $(k + 1)^{th}$ load increment.

The application of advanced iteration schemes using gradients (e.g. Newton–Raphson algorithm) requires the consistent linearization of the nonlinear problem with respect to the basic unknowns summarized in $h$. Using the Newton–Raphson algorithm to solve the nonlinear set of equations (18) involves the solution of a linear system of equations with the Jacobian matrix

$$J = \frac{df}{dh} = I + \tilde{E} \frac{dn}{dh} , \quad (19)$$

being the matrix of coefficients. The treatment of instability phenomena is possible, if appropriate solution procedures like the arc-length method are adopted.

The application of nonlinear boundary conditions (e.g. pressure boundary conditions, traction vector rotating with the normal vector of the boundary during the deformation) lead to an additional term in the nonlinear set of equations (16)

$$0 = f = h + \tilde{E}n - \tilde{C}y - h(\lambda) \quad ; \quad \tilde{C} = \tilde{H}A^{-1} \tilde{B} + \tilde{G} , \quad (20)$$

which accounts for the influence of the nonlinear part $y^{(n)}$ of the prescribed boundary condition $y = y^{(l)} + y^{(n)}$.

For incompressible materials, the unknown pressure $p$ (see eqn (7)) is determined by adding the incompressibility conditions to the nonlinear set of equations (16) or (20), respectively.
4 Computational aspects

In former implementations of the boundary-domain element method, the bottleneck has been the solution of the linear set of equations within the Newton-Raphson algorithm, which has been performed by a LU decomposition of the Jacobian matrix (eqn (19)) and a subsequent forward/back substitution. In the current implementation, the computational effort for this step could be decreased dramatically by using the generalized minimum residual method (GMRES). In a typical elastoplastic problem with 2355 degrees of freedom, the computation time of the GMRES algorithm ranges from only 17 to 45 seconds (31–87 iterations) to solve one linear set of equations, compared to 2256 seconds for the classical LU decomposition of the Jacobian matrix. Further improvements may be achieved, if only the most significant part of the diagonal dominant Jacobian matrix is computed and stored. In this case, only an approximate correction is obtained in each iteration of Newton’s method, which may result in a greater number of iterations. But the final result will be accurate, because the computation of the residual vector \( \mathbf{f} \) (eqns (16), (18) and (20)) is not influenced.

Another bottleneck is the amount of memory that is required to store the fully populated matrix \( \hat{\mathbf{E}} \) (eqn (17)). In contrast to the Jacobian matrix \( \mathbf{J} \), it is not possible to use a sparse approximation of the matrix \( \hat{\mathbf{E}} \) for the computation of the residual error \( \mathbf{f} \). When comparing degrees of freedom of the finite element method (FEM) with those of the boundary-domain element method, it must be taken into consideration, that the FEM matrices are banded, while the BEM matrices are fully populated.

If the area where large deformations occur is small, like e.g. the bending of a notched bar with its end mainly performing a finite rigid body rotation, one collocation node is sufficient for those parts where the deformation is small, but which perform a finite rotation.

5 Examples

5.1 Snap-through of an axisymmetric structure

In this section, the snap-through of a rubber-like axisymmetric structure (see Fig. 1) under internal pressure is considered. The stress relation \( S_{ij} = \mu(1 + \beta)\delta_{ij} - \mu(1 - \beta)(C^{-1})_{ik}(C^{-1})_{kj} - p(C^{-1})_{ij} \) for
Figure 1: Discretization and deformation due to internal pressure.

Figure 2: Deflection in dependence on the internal pressure.
a Mooney–Rivlin rubber-like material is used as hyperelastic constitutive law. The unknown hydrostatic pressure $p$ must be determined from the condition of incompressibility. The material parameters are chosen as $\mu = 100$ MPa and $\beta = 0.25$.

The boundary of the FBEM model was discretized by 56 quadratic boundary elements and the domain by 75 eight-noded quadrilateral cells, as shown in Fig. 1. The axisymmetric structure is supported at the upper boundary (displacement $u_z = 0$), and the pressure $\lambda p_0$ is given at the inner surface. The pressure boundary condition means, that the prescribed tractions $t_i = \frac{\partial a}{\partial A} t_i = -\frac{\partial a}{\partial A} \lambda p_0 n_i$ are not known in advance, but change their direction and magnitude as the structure is deformed.

The deflection of point A in dependence on the load factor $\lambda$ is shown in Fig. 2. When the deformation of the structure is controlled by the pressure as load parameter, the deflection initially increases as $\lambda$ is raised. After reaching the pressure corresponding to the load factor $\lambda_1$, a sudden snap through of the structure along the dashed line could be observed in an experiment. Finally a stable position on the lower branch would be reached. Using the prescribed pressure as load parameter, the problem could not have been solved beyond $\lambda_1$ with standard numerical procedures. But the unstable equilibrium path between $\lambda_1$ and $\lambda_2$ can be traced numerically, if an arc-length method is used.

### 5.2 Elastoplastic deformation of a notched bar

The elastoplastic elongation of the axisymmetric, notched rod shown in Fig. 3 was computed in 30 load steps. The constitutive model proposed by Simo [1, 2], which is based on a Neo-Hookean hyperelastic relation, the von Mises yield criterion and the principle of maximum plastic dissipation, has been used. The material parameters are: shear modulus $\mu = 80194$ MPa, bulk modulus $K = 164206$ MPa, linear hardening $h = 300$ MPa, yield stress $\kappa_0 = 200$ MPa, yield stress $\kappa_\infty = 600$ MPa, saturation coefficient $\delta = 16$. The problem was discretized with 60 quadratic boundary elements and 152 eight-noded domain cells resulting in 2505 degrees of freedom.

The influence of different boundary and domain discretizations and the comparison with finite element results will be discussed in detail in the presentation.
Figure 3: 10% elongation of an axisymmetric rod with a circular notch: geometry, deformed mesh, contours of equivalent Mises stress.

References


