Regularizing transformation method for evaluation of singular and near-singular integrals
M. Kathirkamanayagam, J.H. Curran and S. Shah
Department of Civil Engineering, University of Toronto
Toronto, Ontario, Canada M5S 1A4
Email: curran@ecf.utoronto.ca

Abstract

Regularization based on the method of degenerate mapping is used to evaluate weakly singular and near-singular integrals over triangular domains which arise in the boundary element method. This mapping transforms an integral over a triangular domain to an integral over a square and thus reduces the order of the singularity by one. A procedure which converts the domain integral to non-singular line integrals is given. The proposed method has the advantage of using global coordinates and is easy to implement.

1 Introduction

Homogeneous isotropic continuum mechanics problems form a major category of boundary element applications. A considerable amount of computational time in the boundary element method (BEM) is spent on the evaluation of singular and near-singular integrals.

The singular and near-singular integrals arising in three-dimensional homogeneous isotropic problems are of the form

\[ \iint_\Delta \sum_p c_p \frac{x^k y^l z^m}{r^n(X_o, X)} ds \]  

where \( X_o \equiv (x_o, y_o, z_o) \) is the field point, \( X \equiv (x, y, z) \in \Delta \) is the load point, \( \Delta \) is the domain of integration and

\[ r(X_o, X) \equiv \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2} \]

is the Euclidean distance. The integral is singular if \( X_o \in \Delta \), i.e., the field point lies in
the domain of integration. Near-singular integrals arise when the field point is situated very close to the domain of integration.

The most elementary discretization in the 3-D BEM is the flat triangular subdivision of the boundary. Triangular boundary elements are also the most convenient and widely used in BEM. Hence, in this paper we consider the domain of integration as a planar triangle and derive a computationally efficient scheme for weakly singular and near-singular integrals.

The evaluation of singular integrals encountered in BEM has been the subject of many papers in the past decade. Huang and Cruse\(^1\) published a short review of all integration techniques. The analytical solution over a plane triangular integration domain was first attempted by Cruse\(^2\). Later, Hayami and Brebbia\(^3\) and Cruse and Aithal\(^4\) used polar coordinate mapping to evaluate weakly singular integrals. Vijayakumar and Cormack\(^5\) took a markedly different approach, namely, the invariant imbedding method to evaluate both weakly singular and hyper singular integrals.

Of almost equal importance to the evaluation of singular integrals in BEM is the accurate evaluation of near-singular integrals. Generally, an accurate evaluation of these nominally non-singular integrals is hampered by the rapid increase of the value of the integrand as one moves closer to the proximate singular point i.e., the image of the field point on the integration domain. In most cases, the integrand becomes infinite and is difficult to evaluate numerically. Medina and Liggett\(^6\) and Davey and Hinduja\(^7\) developed analytical closed-form solutions for a class of integrals with simple integration domains. Lachet and Watson\(^8\) and Mustoe\(^9\) adopted special measures to deal with the rapid variation of the integrand in the evaluation of near-singular integrals. An alternative approach was introduced by Vijayakumar and Cormack\(^10\), in which the near-singular integral is considered as a continuation of the singular integral. This idea was used to convert the integral over the domain into an integral along the boundary of the domain, thus avoiding the large numerical variation of the integrand.

In this paper, the method of degenerate mapping is used for the evaluation of near-singular integrals. The degenerate mapping approach was originally introduced by Lachet and Watson\(^8\). Later, it was also used for the evaluation of Cauchy Principal Value (CPV) integrals and Hadamard’s finite part of the hyper singular integrals\(^11,12\) but its validity for these cases was strongly criticized by Huang and Cruse\(^1\). Nevertheless, the degenerate approach is still very useful for the evaluation of weakly singular integrals and near-singular integrals.
A procedure to convert weakly singular and near-singular planar surface integrals into singular and near-singular line integrals is presented. The resulting line integrals can be evaluated using regular numerical methods such as Gaussian quadrature. In addition, the resulting integrals are specified in global coordinates making the results convenient for direct application. As a first step, a degenerate mapping of a triangular integration domain to a square domain is used. Then the domain integral is converted to line integrals using integration by parts.

In section 2, the triangle-to-square degenerate mapping is described. Although several such mappings have been discussed in BEM literature, only the mapping relevant to the present work is discussed.

In section 3, the procedure to convert weakly singular integrals into non-singular line integrals is developed. Since the Jacobian of the degenerate mapping is zero at the singular point, the order of the singularity is reduced by one. By the appropriate choice of coordinate system closed-form integration can be carried out along one of the coordinate axes. The remaining integration reduces to the evaluation of line integrals along the boundary of the domain.

Section 4 outlines the procedure to convert the near-singular integral into non-singular line integrals which require far fewer collocation points for evaluation. First we translate the coordinate system such that one of the vertices of the triangular integration domain becomes the new origin. Then, the degenerate mapping is applied in this translated coordinate system. Using closed-form integration a line integral is obtained. However, the resulting non-singular line integral cannot be determined analytically as in the case of singular integrals. A Gaussian quadrature scheme is used to evaluate the line integrals.

2 Triangle-to-Square Degenerate Mapping

The family of triangle-to-square degenerate mappings has been known for a long time by mathematicians. However it was first introduced in BEM literature by Lachet and Watson. In order for this mapping to be effective, the integration domain needs to be triangular and the singular point must be located at one of the vertices. Generally, the singular point may be located at an arbitrary point on the plane of the integration domain. We can identify four categories of relative positions of the singular point as depicted in Figure 1:

- Field point located at a vertex of the triangle
Among the above four categories, only one contains the singular point at a vertex. Hence further subdivision is required so that each integration domain has the singularity at a vertex as shown in Figure 2. After suitable subdivision, each triangle with the singularity at a vertex is mapped to a right angled triangle (Figure 3). We will refer to the target domain of this mapping as the \((\alpha, \beta)\) plane. The Jacobian of this mapping remains regular. The right angle triangle is then mapped to a square with
vertices at the coordinates \((-1, -1), (1, -1), (1, 1), (-1, 1)\). This plane will be referred to as the \((\xi, \eta)\) plane. This transformation maps the singular point to the side 1-2 of the square.

The weakly singular and near-singular integrals satisfy Riemann’s summability condition. Hence, the integral over an element is the sum of the integrals over the subdivided triangles.

Assuming the coordinates of the vertices of the original triangle are \(O(0, 0, 0)\), \(P_2(x_2, y_2, z_2)\) and \(P_3(x_3, y_3, z_3)\), the two-step composite transformation described above gives

\[
x(\xi, \eta) = \frac{1}{2} (1 + \eta) x_\xi
\]  

(2)
\[
y(\xi, \eta) = \frac{1}{2} (1 + \eta) y_\xi
\]  
(3)

\[
z(\xi, \eta) = \frac{1}{2} (1 + \eta) z_\xi
\]  
(4)

**Figure 3: Triangle-to-square degenerate mapping**
where

\[ x_\xi = \frac{1}{2}(x_2 + x_3) + \frac{1}{2}(x_2 - x_3)\xi \]  
\[ y_\xi = \frac{1}{2}(y_2 + y_3) + \frac{1}{2}(y_2 - y_3)\xi \]  
\[ z_\xi = \frac{1}{2}(z_2 + z_3) + \frac{1}{2}(z_2 - z_3)\xi \]  

The Jacobian, \( J(\xi, \eta) \), of this mapping, is given by

\[ J(\xi, \eta) = \frac{A}{4}(1 + \eta) \]  

where \( A \) is the area of the triangle \( O_{P_1 P_2} \).

### 3 Weakly Singular Integrals

The integrand in eqn (1) can be considered as the summation (with appropriate coefficients) of several terms of the form

\[ f(x, y, z) = \frac{x^ky^lz^m}{r^n(X_o, X)} \]

with different values for \( k, l, m \). It is sufficient to consider a single form of the integral without loss of generality as

\[ I = \iint f(x, y, z)ds \]

The order of singularity of the integrand is

\[ \beta = n - k - l - m \]

Applying the degenerate mapping as described in section 2, we obtain

\[ I = \int\int x^k(\xi, \eta)y^l(\xi, \eta)z^m(\xi, \eta)J(\xi, \eta)d\xi \, d\eta \]

\[ r(\xi, \eta) = \frac{1}{2}(1 + \eta)R_\xi \]
and \( R_\xi = \sqrt{x_\xi^2 + y_\xi^2 + z_\xi^2} \). \hspace{1cm} (14)

Using eqns (2) to (8), the integral can be further simplified to

\[
I = \int \int \frac{AF(\xi)}{d^2(1 + \eta)^{\beta-1}} d\xi d\eta \hspace{1cm} (15)
\]

where

\[
F(\xi) = \frac{x_\xi^k y_\xi^l z_\xi^m}{R_\xi^n} \hspace{1cm} (16)
\]

Since the Jacobian of the transformation vanishes at the singular point, the integrand will be non-singular in the transformed coordinate system. In fact, in the case of the weakly singular integral \( \beta=1 \). Hence the integral is given by

\[
I = \int \int \frac{AF(\xi)}{d^2(1 + \eta)^{\beta}} d\xi d\eta \hspace{1cm} (17)
\]

which after simplification becomes

\[
I = \int F(\xi) d\xi \hspace{1cm} (18)
\]

### 4 Near-Singular Integrals

Consider the integral given by eqn (9). We introduce a translational coordinate mapping

\[
x = x_1 + x \hspace{1cm} (19)
\]

\[
y = y_1 + y \hspace{1cm} (20)
\]

\[
z = z_1 + z \hspace{1cm} (21)
\]

In this translated coordinate system \((x, y, z)\), the triangular integration domain \(P_1 P_2 P_3\) becomes OAB with coordinates O(0, 0, 0),
A \( (x_2, y_2, z_2) \) and \( B \( (x_3, y_3, z_3) \). The coordinates of the field point become \( H(-x_1, -y_1, -z_1) \).

The composite degenerate transformation is given by

\[
\begin{align*}
\tilde{x} &= \frac{1}{2} (1 + \eta) x_{\xi} \\
\tilde{y} &= \frac{1}{2} (1 + \eta) y_{\xi} \\
\tilde{z} &= \frac{1}{2} (1 + \eta) z_{\xi}
\end{align*}
\]

where

\[
\begin{align*}
\tilde{x}_{\xi} &= \frac{1}{2} (x_2 + x_3) + \frac{1}{2} (x_2 - x_3) \xi \\
\tilde{y}_{\xi} &= \frac{1}{2} (y_2 + y_3) + \frac{1}{2} (y_2 - y_3) \xi \\
\tilde{z}_{\xi} &= \frac{1}{2} (z_2 + z_3) + \frac{1}{2} (z_2 - z_3) \xi
\end{align*}
\]

As in the case of the weakly singular integral, after substitution using eqns (22) to (27) the integral given by

\[
I = \frac{A}{4} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \frac{(x_1 + \tilde{x})^k (y_1 + \tilde{y})^l (z_1 + \tilde{z})^m (1 + \eta)^n d\xi d\eta}{r^n}
\]

can be simplified to

\[
I = \frac{A}{4} \int_{-1}^{1} \int_{-1}^{1} \sum_{v=0}^{k+l+m} \frac{c_v (\tilde{\xi})^n (1 + \eta)^{(1+v)}}{(\frac{1}{4} R_{\xi}^2 (1 + \eta)^2 + d_{\xi} (1 + \eta) + \rho^2)^{n/2}} d\xi d\eta
\]

where

\[
\begin{align*}
\tilde{R}_{\xi}^2 &= (\tilde{x}_{\xi}^2 + \tilde{y}_{\xi}^2 + \tilde{z}_{\xi}^2) \\
d_{\xi} &= (x_1 \tilde{x}_{\xi} + y_1 \tilde{y}_{\xi} + z_1 \tilde{z}_{\xi}) \\
\rho^2 &= x_1^2 + y_1^2 + z_1^2
\end{align*}
\]
The functions $c_{\nu}(\xi)$ are the coefficients of the terms $(1 + \eta)\nu$ in the expansion using eqns (22)-(27) of the expression

$$E(\xi, \eta) = (x_1 + x)^k (y_1 + y)^l (z_1 + z)^m.$$  

(33)

4.1 Example

Consider the integral

$$I_3 = \iiint_{\Delta} \frac{1}{r^3} ds$$  

(34)

Using the results given by eqn (29) we get

$$I_3 = \frac{A}{4} \iint_{-1}^{+1} \frac{(1 + \eta)}{(1 + \eta)^2 + R^2 (1 + \eta)^2 + d^2 (1 + \eta) + \rho^2} d\xi d\eta$$  

(35)

After integrating with respect to $\eta$ we obtain

$$I_3 = A \int_{-1}^{+1} \frac{(\rho - d + \rho^2)}{R^2 \rho^2 - d^2 [((x_1 + x)^2 + (y_1 + y)^2 + (z_1 + z)^2] d\xi}$$  

(36)

This line integral can be evaluated using Gaussian quadrature.

5 Conclusions

In this paper, line integration formulae are derived for the singular and near-singular integrals that arise in the boundary element method. The resulting line integrals of the weakly singular integrals are integrated analytically. For the near-singular integrals, numerical integration techniques such as Gaussian quadrature are required. Since the results are expressed in the original global coordinate system, they can be directly utilized in boundary element applications.
Acknowledgments

The financial support of the Natural Sciences and Engineering Research Council of Canada (operating grant - J.H. Curran) and Rocscience Inc. is gratefully acknowledged. The authors would also like to thank Sinnathurai Vijayakumar for his insightful comments in reviewing this paper.

References

258 Boundary Elements


