Dynamic fracture mechanics with crack edges contact interaction

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Abstract

This paper presents some elastodynamic contact problems with unilateral restrictions and friction for bodies with cracks. There are considered case of general dynamic loading and also relevant case of harmonic loading. The mathematical aspects of those problems are discussed briefly.

The problems of a tension-compression plane harmonic wave interaction with one and two co-linear cracks of the finite length with allowance of unilateral contact interaction of the crack edges are solved. The influence of contact interaction of the cracks edges on a stress intensity factor is worked out.

1. Statement of problem

Let we have an elastic body in three-dimensional Euclidean space \( \mathbb{R}^3 \) that occupies the volume \( V \). The boundary of the body \( \partial V \) is a piece wise-smooth one and consists of the parts \( \partial V_p \) and \( \partial V_u \), where the vectors of surface load \( p(x,t) \) and displacements \( u(x,t) \) are assigned respectively. There is an arbitrary oriented crack in the body, which is described by its surfaces \( \Omega^+ \) and \( \Omega^- \). The body may by affected upon by body forces \( b(x,t) \). Its stress-strain state is described by the equations of the linear elastodynamic in displacements Guz' and Zozulya [4]

\[
A_{ij} u^j + b_i = \rho \frac{\partial^2}{\partial t^2} u_i , \quad \forall x \in V , \quad \forall t \in [t_0, T]
\]

The operator \( A_{ij} \) for an isotropic body has the form

\[
A_{ij} = \mu \delta_{ij} \partial_k \partial_k + (\lambda + \mu) \partial_i \partial_j ,
\]
where \( \partial_{x_i} = \frac{\partial}{\partial x_i} \) and \( \partial_{t} = \frac{\partial}{\partial t} \) are derivatives with respect to the space coordinate and time respectively, \( \lambda \) and \( \mu \) are the Lame constants, \( \rho \) is the density of the material. The summation convention applies to repeated indices.

For the correct formulation of the elastodynamic problems it is necessary to assign the initial and boundary conditions. We present these in the form

\[
\begin{align*}
\mathbf{u}_i(x,t_0) &= \mathbf{u}_i^0(x), \quad &\partial_{t} \mathbf{u}_i(x,t_0) &= \mathbf{v}_i^0(x), \quad \forall \mathbf{x} \in \mathcal{V} \\
\mathbf{p}_i(x,t) &= \sigma_{ij}(x,t)n_j(x) = \psi_i(x,t), \quad &\forall \mathbf{x} \in \partial \mathcal{V}_p, \quad &\forall t \in \mathcal{T} \\
\mathbf{u}_i(x,t) &= \varphi_i(x,t), \quad &\forall \mathbf{x} \in \partial \mathcal{V}_u, \quad &\forall t \in \mathcal{T}
\end{align*}
\]

Let us formulate the conditions, that must be satisfied on the crack edges. For the contact forces of interaction and displacements discontinuity vectors the one sided restrictions with friction in the form of inequalities on the edges of the cracks must be satisfied Duvaut and Lions [3], Guz' and Zozulya [4], Panagiotopoulos [7]

\[
\begin{align*}
\Delta \mathbf{u}_n \geq h_0, & \quad q_n \geq 0, \quad (\Delta \mathbf{u}_n - h_0) q_n = 0, \quad \forall \mathbf{x} \in \Omega^+, \quad \forall t \in \mathcal{T} \\
|q_\tau| \leq k_\tau q_n \Rightarrow \partial_{\tau} \Delta \mathbf{u}_\tau = 0, & \quad |q_\tau| = k_\tau q_n \Rightarrow \partial_{t} \Delta \mathbf{u}_\tau = -\lambda_\tau q_\tau
\end{align*}
\]

where \( q_n, q_\tau, \Delta \mathbf{u}_n, \Delta \mathbf{u}_\tau \) are the normal and tangential components of contact forces and displacements discontinuity vectors respectively, \( h \) is the initial opening of cracks, \( k_\tau \) and \( \lambda_\tau \) are coefficients dependent on the contacting surfaces properties.

Therefore, the elastodynamic contact problem for the body with cracks is reduced to the solution of the initial-boundary problem (1), (2) with restrictions in the form of inequalities (3).

The Laplace transformation with respect to the time is widely used for the solution of the elastodynamic problems. We have demonstrated, that this method is effective in the solution of the elastodynamic unilateral contact problems for bodies with cracks Guz' and Zozulya [4,5].
2. Boundary integral equations and algorithm of the problem solution

We use the direct formulation of the boundary integral equations method in the Laplace transformation space for the solution of the elastodynamic contact problems with unilateral restrictions for the bodies with cracks. In principle, everything will stated in regard to the boundary integral equations method in the Laplace transformation space takes place also for the case of harmonic loading.

We obtain the integral representations of the surface forces and displacements vectors on the surface of body dV using formulas Somiliano and also the properties of the surface potentials on the boundary and on the surface of cracks Brebbia, Telles and Wrobel [2], Zozulya [12]. On the smooth parts of the boundary this integral representations have the following form

\[
\begin{align*}
\frac{1}{2} u_j (x,k) &= \int_{\partial V} p_j (y,k) U_{ji} (y - x,k) dS - \int_{\partial V} u_j (y,k) W_{ji} (y,x,k) dS + \\
&\quad \int_{\partial V} f_j (y,k) U_{ji} (y - x,k) dV \\
\frac{1}{2} p_j (x,k) &= \int_{\partial V} p_j (y,k) K_{ji} (y - x,k) dS - \int_{\partial V} u_j (y,k) F_{ji} (y,x,k) dS + \\
&\quad \int_{\partial V} f_j (y,k) K_{ji} (y - x,k) dV
\end{align*}
\]

(4)

The kernels in this formulas are Green's functions of the elastodynamic of the Laplace transformation space Guz' and Zozulya [4].

The integral representations (4) are used in constructing the boundary integral equations initial-boundary-value problems of the elastodynamics in the Laplace transformation space.

The algorithm of the problem solution is reduced to finding the saddle point of subdifferential functional on the sets of one sided restrictions with friction (3). We have used an algorithm of the Udzavy type we developed in Zozulya [9,10,11].

The algorithm is reduced to the following:

a) the initial distribution of the contact forces on the crack edges

\[ p_i^0 (x,t), \ \forall x \in \Omega, \ \forall t \in \mathcal{T} \] is assigned;
b) the Laplace transformation of the contact forces vectors are calculated from the formula

$$p_i^0(x, k) = \mathcal{L}\{p_i^0(x, t)\}, \forall x \in \Omega, \forall k : \text{Re}(k) > \text{Re}(k_0);$$

c) the systems of the boundary integral equations is solved and unknowns on the boundary $p_t(x, k), u_t(x, k)$ and $\Delta u_i(x, k), \forall k : \text{Re}(k) > \text{Re}(k_0)$ are defined;

d) the components of the displacements discontinuities vector on $\Omega$ are calculated from $\Delta u_i(x, k)$ using inverse Laplace transformation

$$\Delta u_i^1(x, t) = \mathcal{L}^{-1}\{\Delta u_i^1(x, k)\}, \forall x \in \Omega, \forall k : \text{Re}(k) > \text{Re}(k_0);$$

e) the normal and tangential components of the contact forces vector are corrected in such way as to satisfy the restrictions (3)

$$p_n^1(x, t) = P_n\{p_n^0(x, t) - \rho_n[\Delta u_n^1(x, t) - h_0(x)]\}$$

$$p_t^1(x, t) = P_t\{p_t^0(x, t) - \rho_t \partial_t \Delta u_t^1(x, t)\}$$

were

$$P_n(p_n) = \begin{cases} 0, & \text{if } p_n \leq 0 \\ p_n, & \text{if } p_n > 0 \end{cases}$$

$$P_t(p_t) = \begin{cases} p_t, & \text{if } |p_t| \leq k_t p_n \\ k_t p_n \frac{p_t}{|p_t|}, & \text{if } |p_t| > k_t p_n \end{cases}$$

are operators for orthogonal projection onto the sets $p_n \geq 0$ and $|p_t| \leq k_t p_n$, $\rho_n$ and $\rho_t$ are chosen on basis conditions of the best convergence of the algorithm;

f) then proceed to the second step of the iteration.

The mathematical investigation of this algorithm convergence are studied in Zozulya [10,13]. Other algorithms and there mathematical investigations was studied in Alibadi and Brebbia [1], Duvaut and Lions [3], Kikuchi and Oden [6], Panagiotopoulos[7].

3. The harmonic loading in a plane with cracks

Consider two collinear cracks of the length $l_1$ and $l_2$ in a plane $\mathbb{R}^2 = \{x : x_3 = 0\}$. Their position is determined by coordinates
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\[ \Omega_1 = \{ x : x_2 = 0 , -l_1 \leq x_1 \leq l_1 \} \]
\[ \Omega_2 = \{ x : x_2 = h_2 , h_1 - l_1 \leq x_1 \leq l_1 + h_1 \} \]

where \( h_1 \) and \( h_2 \) are the distances between the centers of the cracks. A harmonic tension-compression wave propagates in the plane. The incident wave is described by a potential function

\[ \psi(x, t) = \psi_0 e^{\frac{i(k_1 n \cdot x - \omega t)}{c_1}} \]

\[ k_1 = \omega/c_1 , c_1 = \sqrt{(\lambda + 2\mu)/\rho} , n = (\cos \alpha, \sin \alpha) \]

where \( \lambda, \mu, \rho \) are Lamé constants and the density of a material, \( \omega = 2\pi T^{-1} \) is the frequency and \( T \) is the period of vibrations, \( \psi_0 \) is the amplitude of the tension-compression wave, \( c_1 \) is its velocity, \( \alpha \) is the angle of the incidence wave.

We will consider the problem for reflected waves. The load on the crack edges caused by the incident wave has the form

\[ p(x, t) = p_0 \begin{cases} e^{i(k_1 x \cos \alpha - \omega t)} , & \forall x \in \Omega_1 \\ e^{i(k_1 [x + h_1] \cos \alpha + h_2 \sin \alpha - \omega t)} , & \forall x \in \Omega_2 \end{cases} \]

\[ \tau(x, t) = \tau_0 \begin{cases} e^{i(k_1 x \cos \alpha - \omega t)} , & \forall x \in \Omega_1 \\ e^{i(k_1 [x + h_1] \cos \alpha + h_2 \sin \alpha - \omega t)} , & \forall x \in \Omega_2 \end{cases} \]

\[ p_0 = -\mu k_2^2 \left[ 1 - 2(c_2/c_1)^2 \cos^2 \alpha \right] \phi_0 , \quad \tau_0 = -\mu k_2^2 (c_2/c_1)^2 \sin^2 \alpha \phi_0 , \quad k_2 = \omega/c_2 , \quad c_2 = \sqrt{\mu/\rho} \]

The contact forces on the cracks edges \( q(q_n, q_\tau) \) and displacements \( \Delta u(\Delta u_n, \Delta u_\tau) \) discontinuity vectors must satisfy the boundary unilateral conditions with friction in the form

\[ \Delta u_n \geq -h_0 , \quad q_n \geq 0 , \quad (\Delta u_n + h_0)q_n = 0 , \forall x \in \Omega , \forall t \in [0, T] \]

\[ |q_\tau| \leq k_\tau q_n \Rightarrow \partial_t \Delta u_\tau = 0 , \quad |q_\tau| = k_\tau q_n \Rightarrow \partial_t \Delta u_\tau = -\lambda_\tau q_\tau \quad (6) \]
Here we use notation from (3).

Taking into account the contact interaction of the cracks edges the load vector on the crack edges has the form

\[ \mathbf{p} = (p_1, p_2), \quad p_2 = p_0 + q_n, \quad p_1 = \tau_0 + q_\tau, \quad q = 0, \quad \forall x \notin \Omega_c \]

where \( \Omega_c \) is a region of complete contact.

The presence of unilateral restrictions (6) makes the problem non-linear. Therefore the problem for reflected waves is described by the periodic, but not the harmonic functions. The components of the stress-strain state due to the reflected waves cannot be presented as functions of the coordinates multiplied by \( e^{-i\omega t} \) as it is usually done in the solution of elastodynamics problems connected with harmonic loading. In the problem under consideration we will present the components of the displacement vector and stress tensor by Fourier series in the load parameter \( \omega \)

\[ u_\alpha(x,t) = \text{Re} \left\{ \sum_{-\infty}^{\infty} u_{\alpha}^k(x) e^{i\omega_k t} \right\}, \quad \sigma_{\alpha\beta}(x,t) = \text{Re} \left\{ \sum_{-\infty}^{\infty} \sigma_{\alpha\beta}^k(x) e^{i\omega_k t} \right\} \]

where \( \omega_k = \omega k \), and

\[ u_{\alpha}^k(x) = \frac{\omega}{2\pi} \int_0^T u_\alpha(x,t) e^{i\omega_k t} dt, \quad \sigma_{\alpha\beta}^k(x) = \frac{\omega}{2\pi} \int_0^T \sigma_{\alpha\beta}(x,t) e^{i\omega_k t} dt \]

The Fourier coefficients \( u_{\alpha}^k(x) \) and \( \sigma_{\alpha\beta}^k(x) \) are expressed by the Fourier coefficients of displacements discontinuity \( \Delta u_{\alpha}^k(x) \) in the form

\[ \begin{align*}
  u_{\alpha}^k(x) &= - \int_{\Omega} W_{\beta\alpha}(y,x,\omega_k) \Delta u_{\beta}^k(y) d\Omega \\
  \sigma_{\alpha\beta}^k(x) &= - \int_{\Omega} \phi_{\alpha\beta\gamma}(y,x,\omega_k) \Delta u_{\gamma}^k(y) d\Omega
\end{align*} \]

(7)

In the same way the surface load on the crack edges and it’s opening may be presented by their Fourier series.
The Fourier coefficients of the surface forces $p^k(x)$ and displacements discontinuity $\Delta u^k(x)$ vectors are connected by the system of integral equations

$$
\mathbf{p}(x,t) = \text{Re} \left\{ \sum_{-\infty}^{\infty} p^k(x)e^{i\omega_k t} \right\}, \quad \Delta \mathbf{u}(x,t) = \text{Re} \left\{ \sum_{-\infty}^{\infty} \Delta u^k(x)e^{i\omega_k t} \right\}
$$

where

$$
p^k(x) = \frac{\omega}{2\pi} \int_{0}^{T} \mathbf{p}(x,t)e^{i\omega_k t} dt, \quad \Delta u^k(x) = \frac{\omega}{2\pi} \int_{0}^{T} \Delta \mathbf{u}(x,t)e^{i\omega_k t} dt
$$

The Fourier coefficients of the surface forces $p^k_\alpha(x)$ and displacements discontinuity $\Delta u^k_\alpha(x)$ vectors are connected by the system of integral equations

$$
p^k_\alpha(x) = -\int_{\Omega} F_{\beta\alpha}(y,x,\omega_k) \Delta u^k_\beta(y) d\Omega, \quad k = 0, \pm 1, \pm 2, ..., \pm \infty \quad \quad (8)
$$

The kernels $W_{\beta\alpha}(y, x, \omega_k), F_{\beta\alpha}(y, x, \omega_k)$ and $\phi_{\beta\alpha\gamma}(y, x, \omega_k)$ in (7) and (8) are the fundamental solutions of the elastodynamic theory in the space of Fourier series. The analytical formulas for that fundamental solutions calculations are presented in Guz' and Zozulya [4].

The approach presented here was applied to some problems with the harmonic loads on the crack edges and with allowance of their contact interaction. See for example Guz' and Zozulya [4,5], Zozulya [8-13].

4. Numerical examples

We will consider here some numerical examples. For simplify computations and analysis of the results, we introduce dimensionless coordinates $x=x_1/l$ and $y=x_2/l$ and frequency wave number $k^* = \omega l/c_1$. In the numerical examples belong the mechanical properties of a material are: elasticity modulus $E=200\text{GPa}$, Poisson ratio $\nu=0.3$ and specific density $\rho=7800 \text{ kg/m}^3$. Dependence of the calculation accuracy on the number of boundary elements along the crack, the number of time intervals into which the loading period $[0,T]$ is broken up and the number of the Fourier terms was studied in Guz' and Zozulya [4].

As indicated in in Guz' and Zozulya [4] the contact interaction of the crack edges exerts on the stress intensity factor. We investigated this influence in Guz' and Zozulya [4,5], Zozulya [8-12].

Some results of the dimensionless value $K^m_l/p$ calculation for two crack are shown in Fig. 1 and 2.
Boundary Elements

\[ \Psi(x,t) = \Psi_0 e^{i(kx_x - \omega t)} \]

\[ K_{m}^{n} \approx \rho \gamma \pi \ell \]

Figure 1.
In this case the maximum value of the stress intensity factor with and without taking into account contact interaction of the cracks edges may differ by 30% and more.

The results are presented here show, the contact cracks edges interaction must be taken into account in the calculation of the strength of structures by the fracture mechanics methods.
References


