A velocity-vorticity formulation for the numerical simulation of 3D fluid flow by boundary-domain integral method

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Abstract

Three-dimensional numerical solution of fluid dynamics using boundary-domain integral method is considered. A velocity-vorticity formulation of the Navier-Stokes equations is adopted. The kinematic equation is written in its parabolic version.

Key words: Boundary Element Method, Velocity-Vorticity Formulation, Incompressible Viscous Fluid, Three-dimensional Problems

1 Introduction

Bouyancy-driven fluid flow analysis in enclosures has many thermal engineering applications, such as heating and cooling of buildings, the cooling of electronic components and solar energy collectors. In the contemporary design of such devices computational fluid dynamics or better, numerical modelling and simulation of complex fluid flow are of great theoretical and practical importance.

In this paper boundary-domain integral method is presented for the solution of general fluid motion problem which is based on the velocity-vorticity formulation of the Navier-Stokes equations. Particular attention is given to formulate appropriate integral representations for all field functions based on the three-dimensional modified Helmholtz fundamental solution. A lot of attention is given also to find appropriate integration procedure. To optimize the position of Gauss points and to reduce the degree of singularity
the polar transformation is applied. For the kinematic velocity equation the false transient approach is applied to increase the convergency of the proposed numerical algorithm. The large sparse system of nonlinear equations is solved with iterative solver. Results of computer code BEEAS - modul BEMFLOW3D - are compared with benchmark solutions.

2 Governing Equations in Velocity-Vorticity Formulation

The partial differential equations set governing the transport phenomena in incompressible fluid motion represents the basic conservation balances of mass, momentum and energy

\[
\frac{\partial v_i}{\partial x_j} = 0, \quad \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho_0 f_i, \quad \rho_0 c_p \frac{\partial T}{\partial t} = -\frac{\partial q_j}{\partial x_j} \pm I_T + \Phi,
\]

where \( v_i \) is the \( i \)-th instantaneous velocity component, \( x_i \) is the \( i \)-th coordinate, \( D/Dt \) represents the substantial or Stokes derivative, \( p \) is the pressure, \( \tau_{ij} \) is the viscous stress tensor and \( f_i \) stands for the body force, e.g. the gravity \( g_i \). The quantities \( \rho_0, c_p \), and \( I_T \) are respectively the constant fluid mass density, specific isobaric heat capacity and heat source or sink. The variables \( T \) and \( q_j \) stand for temperature and heat flux. The term \( \Phi \) is the Rayleigh viscous dissipation function.

With the vorticity vector \( \omega_i \) representing the curl of the velocity field, e.g. written in the symbolic notation

\[
\omega_i = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}, \quad \frac{\partial \omega_i}{\partial x_i} = 0,
\]

the fluid motion computation scheme is partitioned into its kinematic and kinetic aspect. The kinetics is governed by the vorticity transport equation obtained as a curl of the momentum eq.(2), rendering the statement

\[
\frac{D\omega_i}{Dt} = \frac{\partial \omega_j v_i}{\partial x_j} + \nu_0 \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} + \epsilon_{ijk} \nu_k \frac{\partial F}{\partial x_j},
\]

describing the redistribution of the vorticity vector in fluid flow field.

Applying the curl operator directly to the vorticity definition and using continuity equation for incompressible fluid flow the kinematics can be formulated in the form of vector elliptic Poisson’s equation for the velocity
The equation (6) represents the kinematics of an incompressible fluid motion or the compatibility of the velocity and vorticity fields at a given point in space and time.

In a general fluid motion case the proper boundary condition associated to parabolic kinetic eq.(5) is the vorticity definition written for the boundary of the solution domain, e.g.

\[ \omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j} \text{ on } \Gamma. \]  

(7)

The above statement is essential for the conservation of the solenoidality of the vorticity vector. It requires the solution of the coupled mass conservation problem. The boundary condition assigned to elliptic kinematic eq.(6) is the velocity vector prescribed on the boundary, e.g.

\[ v_i = v_i \text{ on } \Gamma. \]  

(8)

The kinematics given by eq.(6) and the velocity boundary condition eq.(8) cannot assure a solenoidality of the velocity field for an arbitrary vorticity distribution, so this property may be fulfilled only by coupling kinetic and kinematic governing equations. Thus, the solenoidality constrains on velocity and vorticity fields require a coupled iterative solution of the nonlinear system given by eqs.(5) and (6), with the corresponding boundary conditions prescribed by eqs.(7) and (8).

To accelerate the convergency of the coupled velocity-vorticity iterative scheme, the false transient approach is used for the kinematic equation, thus in a solution scheme the parabolic eq.(5) and the following parabolized kinematic eq.(6)

\[ \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{1}{\alpha} \frac{\partial v_i}{\partial t} + e_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0, \]  

(9)

where \( \alpha \) is a relaxation parameter, have to be solved iteratively. It is obvious that the governing velocity equation is exactly satisfied only at the steady state \( (t \to \infty) \), when the time derivative vanishes.

3 Integral Representation of Velocity-Vorticity Formulation

The idea is to combine the finite difference method in time and boundary element method in space. One can consider that each component of the
Boundary Elements

vorticity vector $\omega_i(i = 1, 2, 3)$ is governed by a nonhomogeneous modified Helmholtz eq. (13)

$$\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} - \beta \omega_i + b_i = 0 \quad \text{in} \quad \Omega,$$

subject to the normal and essential boundary conditions. The initial condition is included beside convection, deformation and buoyancy source term in pseudo body force vector

$$b_i = \frac{1}{\nu_0} \frac{\partial}{\partial x_j} \left( -v_j \omega_i + \omega_j v_i + e_{ijk} g_k F \right) + \beta \omega_i,_{F-1},$$

yielding the integral representation for the spatial vorticity kinetics

$$c(\xi) \omega_i(\xi) + \int_{\Gamma} \omega_i \frac{\partial u^*}{\partial n} d\Gamma = \frac{1}{\nu_0} \int_{\Gamma} \left( \nu_0 \frac{\partial \omega_i}{\partial n} - \omega_i v_n + v_i \omega_n + e_{ijk} n_j g_k F \right) u^* d\Gamma +$$

$$\frac{1}{\nu_0} \int_{\Omega} \left( \omega_i v_j - v_i \omega_j - e_{ijk} g_k F \right) \frac{\partial u^*}{\partial x_j} d\Omega + \beta \int_{\Omega} \omega_i,_{F-1} u^* d\Omega,$$

where $u^*$ is the modified Helmholtz fundamental solution, i.e. the solution of equation

$$\frac{\partial^2 u^*}{\partial x_j \partial x_j} - \beta u^* + \delta(\xi, s) = 0$$

and given for the space case by expression

$$u^* = \frac{\beta^{1/4}}{(8\pi^3 r)^{1/2}} K_{1/2}(\sqrt{\beta} r),$$

being $K_{1/2}$ the modified Bessel function of the second kind.

Recognizing that each component of the velocity vector $v_i(i = 1, 2, 3)$ obeys a nonhomogeneous modified Helmholtz eq. (15)

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \beta v_i + b_i = 0 \quad \text{in} \quad \Omega,$$

subject to Dirichlet and Neumann boundary conditions being now the pseudo body force vector equals to expression

$$b_i = e_{ijk} \frac{\partial \omega_k}{\partial x_j} + \beta v_i,_{F-1},$$

the following integral representation for the spatial kinematics can be written in a straightforward manner

$$c(\xi) v_i(\xi) + \int_{\Gamma} v_i \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \left( \frac{\partial v_i}{\partial n} - e_{ijk} \omega_j n_k \right) u^* d\Gamma +$$

$$\int_{\Omega} e_{ijk} \omega_j \frac{\partial u^*}{\partial x_k} d\Omega + \beta \int_{\Omega} v_i,_{F-1} u^* d\Omega.$$
4 Discretization and Numerical Solution

For the numerical approximate solution of various considered field functions, e.g. velocity, vorticity, temperature etc., the corresponding boundary-domain integral representation is written in discretized manner in which the integrals over the boundary and domain are approximated by a sum of integrals over $E$ individual boundary elements and $C$ internal cells, respectively. Next the variation of all field functions or their products within each boundary element or internal cell is approximated by the use of interpolation polynomials $\Phi$ or $\phi$ with respect to boundary or domain and nodal function values. Let the index $n$ refers to the number of nodes in each boundary element or internal cell and also relates to the degree of the respective interpolation polynomials. It should be mentioned that in general the degree of interpolation polynomials on the boundary elements and internal cells can differ, making this numerical technique very flexible. In the discretized equations for the kinematics, vorticity kinetics and energy transport the integrals such as $h, g, d_i, b$ etc. are involved, representing the integration over individual boundary element and internal cell respectively, i.e.

\[
\begin{align*}
    h^n_e &= \int_{\Gamma_e} \Phi^n \frac{\partial u^*}{\partial n} d\Gamma, \\
    g^n_e &= \int_{\Gamma_e} \Phi^n u^* d\Gamma, \\
    d^n_{ci} &= \int_{\Omega_e} \phi^n \frac{\partial u^*}{\partial x_i} d\Omega, \\
    b^n_e &= \int_{\Omega_e} \phi^n u^* d\Omega \quad \text{etc.,}
\end{align*}
\]

which are functions of the geometry and material properties.

In developed computer code BEEAS - modul BEMFLOW3D - 4-nodes linear boundary elements and constant internal cells are used for the computation of integrals mentioned above (figure 1).

To optimize the position of Gauss points and to reduce the degree of singularity the polar transformation is applied. The discretization of integral representations for velocity, vorticity and temperature is obtained as usual.

5 Test Examples

Some preliminary tests have been studied first. One of them is heat conduction in three-dimensional bodies with respect to the influence of thermal
conductivity isotropic behaviour in the solid material. Steady state at discretization of the computational domain with only one internall cell is shown in figure 2.

In second case the results of numerical simulation of entry flow with \( v_x = 1 \), \( v_y = 0 \) and \( v_z = 0 \) are shown. In computational model mesh with \( 5 \times 1 \times 1 \) internal cells is used (figure 3).

Couette flow - flow between two parallel plates - for Reynolds number value equals 10 is studied in the test example No. 3 (figure 4). While bottom plate is fixed, top plate is moving with \( v_x = 10 \), \( v_y = 0 \) and \( v_z = 0 \). All other fluid properties are in accordance with \( Re = 10 \). Mesh consists of \( 8 \times 2 \times 8 \) internal cells.

The last test example, problem of natural convection, is often used for test-
equations of motion and energy are coupled through the buoyancy term introducing additional nonlinearity in the nonlinear system of equations. The determination of the temperature field turns to be most delicate, since the numerical errors in the velocity field are magnified while determining temperatures, what may cause a non-convergency of the procedure.

The evaluated three-dimensional case is shown in figure 5. The fluid motion is due to heating on the rear wall, while on the front wall a relative zero temperature was kept. The remained boundaries are assumed to behave
ideally isolated. Rayleigh number value equals 1000.

Figure 5: Natural convection: Discretization and temperature profile, Ra=1000

6 Conclusion

The boundary domain integral method is used to solve three-dimensional viscous fluid flow and energy transport problems. Owing to the spatial modified Helmholtz fundamental solution a part of the transport mechanism is transferred to the solution surface. This procedure should result in a very stable and accurate numerical algorithm. But, the development of computer code is still in progress. There are many difficulties during the evaluation of matrix coefficients. The integration process is not exact enough and it takes a great amount of computational time to compute the coefficients. In developed computer code BEEAS - modul BEMFLOW3D - also the false transient technique for the kinematic velocity equation is used to increase the convergency of the numerical procedure. Comparisons of BEEAS - modul BEMFLOW3D - results with analytical solutions show good agreements.

References

[1] Alujević A., Kuhn G., Škerget L., Boundary Elements for the Solution of


