A Generalized Boundary Integral Equation for Axisymmetric Heat Conduction in Non-Homogeneous Isotropic Media

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Abstract
In this paper, we extend our previous work on the generalized boundary integral equation (BIE) formulation for heat conduction in media with spatially varying thermal conductivity to formulate a generalized boundary integral equation for axisymmetric heat conduction in such media. The kernels of the integral equation do not contain elliptic integrals encountered in traditional BEM formulation in which Dirac Delta forcing functions are used to drive the axisymmetric adjoint equation. The resulting integral equation is implemented in a BEM code. Numerical examples are presented to validate the formulation. Quadratic elements are used and hollow and solid bodies are considered. Results are presented for the case of constant thermal conductivity and spatially varying thermal conductivity.

1 Introduction

Heat conduction problems in non-homogeneous media has been the subject of much research in the BEM community. Mathematical transformations have been proposed[1] along with Dual Reciprocity[2] and perturbation methods[3]. New Green's functions have been discovered for the cases of linearly varying thermal conductivities[4-6]. All previously mentioned methods impose some restriction on the nature of the thermal conductivity variation. Recently, the authors have proposed a new generalized boundary integral formulation for steady heat conduction in heterogeneous isotropic[7] and anisotropic media[8] as well as for transient heat conduction in such cases[9]. Two- and three-
In this paper, we extend the generalized BEM formulation for heat conduction problems with arbitrarily varying thermal conductivity to address the axisymmetric case. Again, a radially-symmetric generalized fundamental solution is sought as a response of the adjoint equation to a radially non-symmetric forcing function at the source point. The latter function is used in place of the traditional Dirac delta function. The integral equation which was derived for the general case of arbitrarily varying thermal conductivity is shown to be directly applicable for the axi-symmetric case and is also shown to be readily applicable to the non-homogeneous axisymmetric case. Three numerical examples are provided to verify the formulation.

2 Generalized boundary integral equation for heat conduction in isotropic non-homogeneous media

In order to arrive at an axisymmetric boundary integral equation, it is necessary to first briefly review the generalized boundary integral equation for steady-state heat conduction in isotropic non-homogeneous media. The governing equation of interest is

$$\nabla \cdot [k(x)\nabla T(x)] = 0$$  \hspace{1cm} (1)

where $k(x)$ is the spatially varying thermal conductivity and $T(x)$ is the temperature. The independent variable $x$ depends on the spatial dimension of the problem and the coordinate system of choice. The above variable coefficient partial differential equation is converted to an integral equation by first introducing a function $E(x, \xi)$ and integrate the product of the governing equation and $E(x, \xi)$ over the domain $\Omega$ of the problem

$$\int_{\Omega} \{E(x, \xi) \nabla \cdot [k(x)\nabla T(x)] \} d\Omega = 0$$  \hspace{1cm} (2)

Using Green's first identity twice, the following integral equation is derived,

$$\int_{\Gamma} \left[ E(x, \xi) k(x) \frac{\partial T(x)}{\partial n} - T(x) k(x) \frac{\partial E(x, \xi)}{\partial n} \right] d\Gamma(x)$$  \hspace{1cm} (3)

$$+ \int_{\Omega} \{T(x) \nabla \cdot [k(x)\nabla E(x, \xi)] \} d\Omega = 0$$

Here, the domain boundary is denoted by $\Gamma$. The dimension of $\Gamma$ is dictated by the dimension of $\Omega$, i.e., if $\Omega$ is 2-D, then $\Gamma$ is 1-D. The next step is to introduce a solution to the adjoint equation perturbed by a singular forcing function, $D$, acting at the source point $\xi$. 


\[ \nabla \cdot [k(x) \nabla E(x, \xi)] = -D(x, \xi) \quad (4) \]

For those problems in which the thermal conductivity does not vary with position, \( D(x, \xi) \) is traditionally taken as the Dirac delta function. In this case, the solution of Eq. (4) in an infinite domain is readily derived as the well known Green's free space or fundamental solution to the steady-state heat conduction equation, \( -(1/2\pi k) \ln(r) \). On the other hand, if the Dirac delta function is used to perturb Eq. (4), then, since the Dirac delta function is symmetric about its source point \( \xi \) and since the adjoint equation is a variable coefficient partial differential equation, the fundamental solution must be non-symmetric. This precludes derivation of fundamental solutions for arbitrary variations of \( k \). To overcome this difficulty, in \([7-9]\) the authors introduce a generalized forcing function\([10]\), \( D(x, \xi) \), which is constructed to obey the following properties:

\begin{align}
\text{(a)} \quad \nabla \cdot [k(x) \nabla E(x, \xi)] &= -D(x, \xi) \\
\text{(b)} \quad \int_{\Omega_c} D(x, \xi) \, d\Omega(x) &= 1 \\
\text{(c)} \quad \int_{\Omega} f(x) \, D(x, \xi) \, d\Omega(x) &= f(\xi) A(\xi) \\
\text{(d)} \quad A(\xi) &= \int_{\Omega} D(x, \xi) \, d\Omega(x) \quad (5) \end{align}

In Eq. (5), the domain \( \Omega_c \) is a circular domain centered about the source point \( \xi \), while \( \Omega \) is arbitrary in shape but encloses the source point \( \xi \). Property 5(b) is a normalizing property of the function \( D \). Introducing Eq. 5(a) into Eq. 5(d) and applying the Gauss divergence theorem, the amplification factor is

\[ A(\xi) = -\int_{\Gamma} [k(x) \frac{\partial E}{\partial n}(x)] \, d\Gamma(x) \quad (6) \]

Invoking the sampling property of the \( D \) function and introducing it to Eq. (3),

\[ A(\xi) T(\xi) = \int_{\Gamma} [E(x, \xi) k(x) \frac{\partial T}{\partial n}(x) - T(x) k(x) \frac{\partial E}{\partial n}(x, \xi)] \, d\Gamma(x) \quad (7) \]

The amplification factor, \( A(\xi) \) is evaluated at all points where the temperature is sought, whether on \( \Gamma \) or within \( \Omega \). The integral equation is solved by standard BEM techniques once \( E(x, \xi) \) is determined. Introducing a local polar coordinate system \( r-\theta \) centered about the source point \( \xi = (x_i, y_i) \), see Fig. 1, the authors derive\([7]\) a generalized fundamental solution \( E(r, x_i, y_i) \) in 2-D as

\[ E(r, x_i, y_i) = -\int_r^{2\pi} \frac{dr}{r} \int_0^r k(r, \theta, x_i, y_i) \, d\theta \quad (8) \]
Similarly, introducing a local spherical coordinate system centered about the source point, \((x_i, y_i, z_i)\), and the authors derive\[7\] the following generalized fundamental solution \(E(r, x_i, y_i, z_i)\) for 3-D as

\[
E(r, x_i, y_i, z_i) = - \int \frac{dr}{r^2 \int_0^{2\pi} \int_0^\pi k(r, \theta, \phi, x_i, y_i, z_i) \sin \theta \, d\theta \, d\phi}
\]  \hspace{1cm} (9)

It is noted that the well known 2-D fundamental solution, \(- (1/2\pi k) \ln r\), and 3-D fundamental solution, \(1/4\pi r k\), are retrieved from Eqs. (8) and (9) for homogeneous media with constant conductivity. Examination of the form of the generalized fundamental solution reveals very interesting features. Specifically, considering the 2-D fundamental solution and integrating by parts

\[
E(r, x_i, y_i) = - \left[ \frac{1}{\int_0^{2\pi} k(r, \theta, x_i, y_i) \, d\theta} \right] \ln r - \int \left( \frac{\int_0^{2\pi} \partial k(r, \theta, x_i, y_i) \, d\theta}{\left[ \int_0^{2\pi} k(r, \theta, x_i, y_i) \, d\theta \right]^2} \right) \ln r \, dr
\]  \hspace{1cm} (10)

or, \(E\) has the form

\[
E(r, x_i, y_i) = - F(r) \ln r - \int G(r) \ln r \, dr
\]  \hspace{1cm} (11)

It can readily be seen that the above fundamental solution has two components: a \(\ln r\) singular component which collapses to the response to a Dirac delta forcing function for a medium with constant thermal conductivity, and a second more complicated component which vanishes in the latter case. Usually, \(F(r)\) can readily be found; however, unless the product of \(G(r)\) and \(\ln r\) is integrable, a closed form expression for \(E\) is unavailable. However, the authors have shown that multidimensional polynomial approximation for \(k\) in such cases yield accurate closed form expression for \(E\). With specification of \(k\), the function \(D\) which generates the above solution is found by introducing Eqs. (8)
or (9) into the adjoint equation, Eq. (4). A typical forcing function $D$ is illustrated below in Figure 2.

![Figure 2 Plot of a typical generalized forcing function $D(r, \theta, x_i, y_i)$ near a source point $(x_i, y_i)$ and shown here for the case $k = (1 + x/100)^{3}$.](image)

### 3 Axisymmetric generalized boundary integral equation for non-homogeneous media

In the case of axisymmetry (geometry, boundary conditions, and material property), the governing heat conduction equation becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r k(r, z) \frac{\partial T}{\partial r} \right] + \frac{\partial}{\partial z} \left[ k(r, z) \frac{\partial T}{\partial z} \right] = 0 \quad (12)$$

In order to avoid confusion between the local polar coordinate system introduced to derive the generalized fundamental solution (see Fig.1) and the global cylindrical coordinate system, we reassign coordinate definitions as follows: the global cylindrical coordinate system will be referred to by using underscores, i.e. $r$ and $z$. In the case of axisymmetry, the domain in Fig. 1 represents the generating curve in the $r - z$ plane. Rearranging and using the newly defined symbols for the global cylindrical coordinate system, the above equation becomes

$$\frac{\partial}{\partial r} \left[ k'(r, z) \frac{\partial E}{\partial r} \right] + \frac{\partial}{\partial z} \left[ k'(r, z) \frac{\partial T}{\partial z} \right] = 0 \quad (13)$$

where a new thermal conductivity $k'(r, z)$ is defined as

$$k'(r, z) = \tau k(r, z) \quad (14)$$

This equation is exactly the same as the 2-D governing equation in Cartesian coordinates for heat conduction in non-homogeneous media with $k'$ as the spatially varying thermal conductivity. Consequently, developments of Eq. (1) - (11) above apply for the axisymmetric case with an appropriate redefinition of the thermal conductivity given in Eq. (14). Thus, the axisymmetric generalized
fundamental solution for non-homogeneous media is readily expressed as

\[ E(r, \xi_i, \bar{z}_i) = - \int_{r_{i}}^{2\pi} \frac{d\theta}{r} \int_{0}^{2\pi} k(r, \theta, \xi_i, \bar{z}_i) \, d\theta + r^2 \int_{0}^{2\pi} k(r, \theta, \xi_i, \bar{z}_i) \cos \theta \, d\theta \]  

(15)

with \( \theta \) defined with respect to a branch cut parallel to the \( r \) axis, and the BIE is

\[ A(\xi_i, \bar{z}_i) T(\xi_i, \bar{z}_i) = \oint_{\Gamma} \left[ E(r, \xi_i, \bar{z}_i) \, r \, k(r, \bar{z}) \frac{\partial T}{\partial n} (r, \bar{z}) \right. \\
- T(r, \bar{z}) \, r \, k(r, \bar{z}) \frac{\partial E}{\partial n} (r, \xi_i, \bar{z}_i) \right] d\Gamma(r, \bar{z}) \]  

(16)

where,

\[ A(\xi_i, \bar{z}_i) = \oint_{\Gamma} \left[ r \, k(r, \bar{z}) \frac{\partial E}{\partial n} (r, \xi_i, \bar{z}_i) \right] d\Gamma(r, \bar{z}) \]  

(17)

Here \((\xi_i, \bar{z}_i)\) is the location of the source point (origin of the local \( r-\theta \) polar coordinate system defined in Fig. 1). The above equation is directly analogous to the 2-D BIE, and, in fact, the 2-D code is directly applicable with the minor modification of Eq. (14). It is clear that the new generalized fundamental solution does not contain the usual elliptic integrals which appear in the traditional BEM formulation for the case of constant thermal conductivity. Further, this approach is general in that it also addresses the inhomogeneous axisymmetric case. It should be noted, however, that care must be taken when applying the above formulation for solid axisymmetric bodies. A problem may arise when the collocation point lies on the axis of symmetry as \( \xi_i = 0 \). The first term in the denominator of Eq. (15) is identically zero, while the second term in the denominator may very well integrate to zero. In order to remedy the situation, a subparametric element or discontinuous element should be used as the last connecting element of the generating curve to the axis of symmetry. This effectively avoids placing the collocation point there.

4 Numerical Implementation

Numerical solution of the BIE, Eq. (7), follows standard BEM. The domain boundary, \( \Gamma \), is discretized using \( N \) – boundary nodes, and isoparametric elements are used to model the temperature and the flux on the boundary. We use quadratic isoparametric boundary elements with double nodes to account for discontinuous boundary conditions. Treatment at corners imposed with Dirichlet conditions follows the formulation of Kassab and Nordlund[11]. We use Gauss-Kronrod (\( G_7 - K_{15} \)) based adaptive quadrature routine DQAGS from QUADPACK to integrate the influence coefficients. The BIE becomes,

\[ A(\xi_i) T(\xi_i) + \sum_{j=1}^{N} H_{i,j} T_j = \sum_{j=1}^{N} G_{i,j} \left( k_j \frac{\partial T}{\partial n} \right)_j, \quad i = 1, 2, \ldots N \]  

(18)

and the amplification factor \( A(\xi) \) is evaluated numerically by discretizing the
contour integral in Eq. (6) or by taking the negative of the sum of the off-diagonal $H_{i,j}$ terms for the appropriate row.

5 Examples

Three examples are considered to verify the above formulation. Quadratic elements are used in all examples. In the first example, the case of a homogeneous solid cylinder of material with constant thermal conductivity $k(r, z) = 1$ is considered. The geometry, boundary conditions, and boundary element discretization are shown in Fig. 3. The analytical solution for this case, plotted in Fig. 4, is derived as:

$$T(r, z) = 160 \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{(n\pi)^3} \sin(n\pi z) \frac{I_0(n\pi r)}{I_0(n\pi)} + 100$$  \hspace{1cm} (19)

Although not apparent in Fig. 3, discontinuous quadratic elements were used
to join the generating curve to the axis of symmetry in this case. The BEM computed solution along with the relative percent error are provided above in Figs. 5 and 6. Excellent agreement is found between the exact and BEM computed solutions with a maximum relative error of 0.2%.

In the second example, a homogeneous hollow cylinder with \( k(r, z) = 1 \), illustrated in Fig. 7, is again considered. Using first kind boundary conditions displayed in Fig. 7, the exact solution, see Fig. 8, is derived as:

\[
T(r, z) = 20 \sum_{n=1}^{\infty} \left[ 1 - (-1)^n \right] \frac{\sin(n\pi z)}{n\pi} \times \frac{K_0(2n\pi)I_0(n\pi r) - I_0(2n\pi)K_0(n\pi r)}{K_0(2n\pi)I_0(n\pi) - I_0(2n\pi)K_0(n\pi)} + 100
\]  

The BEM computed isotherms are displayed in Fig. 9, while the relative percent error is provided in Fig. 10. Examination of the results reveals that the proposed algorithm faithfully reproduces the temperature field.
Next, the irregular hollow inhomogeneous region with linearly varying thermal conductivity $k(r, z) = 5 + z$ illustrated in Fig. 11 is chosen as a final test. An exact solution which satisfies the governing equation, Eq. (12), is taken as

$$T(r, z) = z^2 - r^2 + 10z + 100$$  \hspace{1cm} (21)

This temperature is used to impose the mixed boundary conditions as shown in Fig. 11. The exact solution is provided in Fig. 12, while the BEM computed isotherms and relative percent error distribution are given in Figs. 13 and 14 respectively. The agreement between the BEM computed temperatures and the exact temperatures (within 0.2% deviation) for this mixed boundary condition problem with variable thermal conductivity is a further validation of the method presented in this paper. This concludes successful verification of our method.

Figure 11. Geometry, boundary conditions, and discretization.  
Figure 12. Exact isotherms.  
Figure 13. BEM computed isotherms.  
Figure 14. Relative percent error.
6 Conclusions

The generalized boundary integral formulation previously developed by the authors for heat conduction in non-homogeneous media is extended to address axisymmetric problems. The generalized axisymmetric fundamental solution is shown not to contain the usual elliptic integrals which arise in standard BEM formulation of axisymmetric problems. Further, the formulation is general and includes the case of non-homogeneous axisymmetric heat conduction. The resulting boundary integral equation is discretized using standard BEM. Numerical examples provide verification of the formulation.

7 References