Conformal mapping solutions for the 2D heterogeneous Helmholtz equation

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Abstract: The two dimensional Laplacian operator is maintained in a conformal mapping except for a scaling factor equal to the Jacobian of this transformation, although clearly the two dimensional Laplace equation is unchanged. This allows the use of conformal mapping to obtain Green's functions for two dimensional heterogeneous Helmholtz as well as potential and advective-diffusive, equations with a heterogeneity related to this scaling factor. Such solutions are useful in their own right and in addition can serve as a basis for developing Green's functions for use as kernels in boundary integral/element methods used for numerical solution of complex physical problems.

Introduction: The use of conformal mapping techniques for the solution of problems of mechanics is not new. For example, they have been applied extensively in the past by Soviet elasticians in studies of stress analysis, e.g. Muskhelishvili1, and stress concentration, e.g. Savin2. In these the key idea was to use conformal mapping to reduce a complicated geometry of the original problem to a simple standard geometry in the complex plane without significantly altering the governing equations. Similar applications are widely used in potential problems, e.g. Hromadka3. Such mappings were of course restricted to two dimensional geometries. It does not appear to be widely recognized that these same techniques can be used to reduce several forms of second order partial differential equations with position dependent material properties governing a variety of physical processes in various fields of mechanics.

This study focuses on the two dimensional Helmholtz equation with a position dependent wave number, k(x, y). Such problems arise in i) plane acoustic waves in heterogeneous media, ii) plane electromagnetic waves, e.g. where the electric field is polarized in a plane and iii) horizontally polarized
(SH) elastic shear waves or iv) two dimensional P and SV elastic waves in heterogeneous media, e.g. Achenbach\(^4\), all for time harmonic behavior.

The result is that the same fundamental solution is obtained in the "mapped" domain for all material behaviors but with the argument of that solution dependent on the nature of the heterogeneity. This is in contrast to previous work which uses algebraic transformations, e.g. Shaw and Makris\(^5\), Manolis and Shaw\(^6\), etc. where the form of the fundamental solution has changed but the arguments remain the original ones. These fundamental solutions may then be used directly in existing boundary element methods, e.g. Brebbia, Telles and Wrobel\(^7\), forming a useful addition to existing methodologies for heterogeneous media wave problems, e.g. Brekhovskikh and Beyer\(^8\).

**Methodology:** Consider a standard heterogeneous Helmholtz equation in two dimensions,

\[
\nabla^2 U(x, y) + k^2(x, y) U(x, y) = - Q(x, y)
\]

where \(k\) is a position dependent wave number and \(Q\) is a volume source. In general, Green's functions for such equations are not available and thus the standard boundary integral equation formulation may not be developed. It is of interest to note however that a conformal mapping of \(z = x + i y\) to \(Z = X + i Y\) in the form \(Z = f(z)\) allows the Laplacian operator to be written as

\[
\nabla_{x,y}^2 U(x, y) = J(X(x, y), Y(x, y); x, y) \nabla_{X,Y}^2 U(X, Y)
\]

such that a Laplace equation on \(U(x, y)\) transforms to a Laplace equation on \(U(X, Y)\) as long as the Jacobian, \(J(X(x, y), Y(x, y); x, y)\), of the transformation is not zero, (see the appendix). Transforming eq. [1] by such a mapping leads to

\[
J(X, Y; x, y) \nabla_{X,Y}^2 U(X, Y) + k^2(x, y) U(X, Y) = - Q(x, y)/J(X, Y; x, y)
\]

which in turn leads directly to

\[
\nabla^2 U(X, Y) + k^2_0 U(X, Y) = - Q(x, y)/J(X, Y; x, y)
\]

with \(k^2_0\) a constant and

\[
k^2(x, y) = k^2_0 J(X(x,y), Y(x,y); x, y)
\]

While this is an inverse approach, there clearly are are a wide variety of conformal mappings available for such inverse solutions. When a boundary integral equation is written however, care must be taken to use an appropriate volume source term, i.e. one which makes the right hand side of eq. [4] a delta
function in $(X, Y)$ space in order to be able to write the (homogeneous material) Green's function for $U(X, Y)$. When the boundary integral equation is written in $(x, y)$ space, a coordinate transformation represented by the Jacobian must be included in the two dimension (area) integral, i.e.

$$
\Phi_{(x,y)} \delta(x, y; x_0, y_0) \, dx \, dy = -1 = \Phi_{(X,Y)} \delta(X, Y; X_0, Y_0) \, dX \, dY = \int_{(x,y)} \delta(X, Y; X_0, Y_0) J(X, Y; x, y) \, dx \, dy
$$

and thus $\delta(x, y; x_0, y_0)$ is identified with $\delta(X, Y; X_0, Y_0) / J(x_0, y_0)$. If $Q(x, y)$ is identified with $\delta(X, Y; X_0, Y_0) J(X, Y; x, y)$, in order to have $U(X,Y)$ in Eq.[4] be identified as the homogeneous medium Green's function,

$$
G_{\text{homogeneous}}(X, Y) = (i/4) \frac{1}{|R - R_0|} \quad [7]
$$

then the original equation must also be

$$
\nabla^2 U(x, y) + k^2(x, y) U(x, y) = -Q(x, y) = -\delta(x, y; x_0, y_0) \quad [8]
$$

Thus the Green's function satisfying the original heterogeneous equation is the same form,

$$
G_{\text{heterogeneous}}(x, y) = (i/4) \frac{1}{|R - R_0|} \quad [9]
$$

where $|R - R_0|$ is now equal to $\left[ (X - X_0)^2 + (Y - Y_0)^2 \right]^{1/2}$ and $(X, Y)$ are given through the mapping as functions of $(x, y)$. Some examples are given in the following section. However some care must be taken in discussing the allowable domains in $(x, y)$ as compared to those in $(X, Y)$. The $(X, Y)$ domain is for all space with $X$ and $Y$ both ranging from $-\infty$ to $+\infty$. This does not automatically translate into the same sort of domain for $(x, y)$, although this will be the case for the examples given. However, the transformation of $(x, y)$ into $(X, Y)$ need not be thought of in complex terms but could just as well be some convenient relationship no matter how arrived at.

Example 1: Consider a second power of $z$, (a general form for an arbitrary integer power is given in the appendix), i.e.

$$
Z = f(z) = z^2 \quad [10]
$$

such that

$$
X = x^2 - y^2 \quad ; \quad Y = 2 \, x \, y \quad [11]
$$
This mapping leads directly to $4(x^2 + y^2)$, the Jacobian of this transformation, 
\[ J(x, y) = \frac{\partial (X, Y)}{\partial (x, y)} \], and thus to

\[ \nabla_{x,y}^2 U = 4(x^2 + y^2) \nabla_{X,Y}^2 U \] \hspace{1cm} [12]

Then the heterogeneous Green's function for a material whose wavenumber squared is $k^2(x, y) = 4(x^2 + y^2) k_0^2$ is simply

\[ G_{\text{heterogeneous}}(x, y) = (i/4) H^{(1)}_0(k_0 |R - R_0|) \] \hspace{1cm} [13]

where

\[ |R - R_0| = \left\{ (x^2 - y^2) - (x_0^2 - y_0^2) \right\}^2 + \left\{ (2xy)^2 + (2x_0y_0)^2 \right\}^{1/2} \] \hspace{1cm} [14]

Here an infinite domain for $(X, Y)$ is also an infinite domain for $(x, y)$.

Example 2: Consider the mapping

\[ Z = \exp(z) \] \hspace{1cm} [15]

such that

\[ X = \exp(x) \cos(y) ; Y = \exp(x) \sin(y) \] \hspace{1cm} [16]

The Jacobian of the transformation is exp(2x) and thus the wave number squared satisfies

\[ k^2(x, y) = k_0^2 \exp(2x) \] \hspace{1cm} [17]

Then the fundamental Green's function is given in eq. [13] with

\[ |R - R_0| = \left\{ (\exp(2x) - 2 \exp(x + x_0) + \exp(2x_0)) \right\}^{1/2} \] \hspace{1cm} [18]

Here an infinite domain for $(X, Y)$ is also one for $(x, y)$ but there is no need to consider $y$ to be limited to $(0, 2\pi)$ even though it appears as a sine or cosine function; in fact, $y$ does not actually appear in the final form for the Green's function for this form of heterogeneity. Figures 1 and 2 respectively represent $k(x, y)$ and $G_{\text{heterogeneous}}(x, y)$ for this particular case along with the corresponding functions for the equivalent homogeneous medium case, $k_0$ and $G_{\text{homogeneous}}(x, y)$. The medium uses a reference wavenumber of $k_0 = \pi/2$ (sec$^{-1}$) and a sequence of 30 "receivers" is placed from the origin (source) to a depth of 6 meters in the $y$ direction. An amplification of the amplitude of the heterogeneous Green's function is observed at shallow depths, doubling near to the origin, followed by a reduction in amplitude with increasing distance.
Figure 1: Heterogeneous medium with wavenumber $k(x,y)$.

Figure 2: (a) Amplitude and (b) phase angle of fundamental solution $G(x,y)$ for a heterogeneous medium with wavenumber $k(x,y)$.
from the source. In addition, the phase is distorted due to the continuous scattering brought about by the heterogeneous structure of the medium.

Conclusions: A solution technique has been presented for the two dimensional Helmholtz equation with a particular class of position dependent wavenumber. The technique follows a new path which complements the relatively few other techniques for such problems such as algebraic transformations of the dependent variables, iteration methods based on the homogeneous medium solution and even other independent variable transformations not based on such conformal mappings. There are innumerable mappings available, each corresponding with a particular heterogeneity in the governing Helmholtz equation. They all end with the Green's function as described in eq. [13] but with different definitions of R and possibly some constraints on the domains involved. As discussed by Shaw and Gipson\textsuperscript{9}, these solutions will also lead to Green's functions for heterogeneous media potential and advective-diffusive problems as well.

Appendix:

If \( Z = X + i \ Y = f(z) \) where \( z = x + i \ y \) and \( f \) is an analytic function of \( z \), then \( X \) and \( Y \) obey Cauchy-Riemann conditions,

\[
\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} ; \ \frac{\partial X}{\partial y} = - \frac{\partial Y}{\partial x} \tag{A1}
\]

such that \( X(x, y) \) and \( Y(x, y) \) are harmonic functions, e.g. Churchill\textsuperscript{10}

\[
\nabla^2_{xy} X(x, y) = 0 ; \ \nabla^2_{xy} Y(x, y) = 0 \tag{A2}
\]

It is then relatively simple to show by direct chain rule differentiation that the Laplace operators in the two coordinate systems are related as shown in eq. [2]. Actually, a further step may be taken in relating the heterogeneity to the mapping. The Jacobian is in general

\[
J(X,Y;x,y)=(\partial X/\partial x)(\partial Y/\partial y) - (\partial X/\partial y)(\partial Y/\partial x) = \left[ (\partial X/\partial x)^2 + (\partial X/\partial y)^2 \right] \tag{A3}
\]

so that eq. [5] may be considered in terms of a vector wave number as

\[
\mathbf{k} = k_o \left[ \nabla_{xy} X(x, y) \right] = k_o \left[ (\partial X/\partial x) \hat{i} + (\partial X/\partial y) \hat{j} \right] = \left[ k_x \hat{i} + k_y \hat{j} \right] \tag{A4}
\]

such that \( J = \left[ \mathbf{k} \circ \mathbf{k} \right] / k_o^2 \) and \( X \) is found from

\[
(\partial X/\partial x) = k_x(x,y);(\partial X/\partial y) = k_y(x,y);(\partial Y/\partial y) = k_x(x,y);(\partial Y/\partial x) = - k_y(x,y) \tag{A5}
\]

Then if the wave number can be expressed in this vector form, the \( X \) and \( Y \) may be obtained by integrating two sets of first order differential equations.
In example 1, \( J \) was equal to \( 4(x^2 + y^2) \). In this case, \( k^2 / k_0 \) would be \( 2x \hat{x} + 2y \hat{y} \) which would lead to the correct \( X \) and \( Y \) (to within an arbitrary constant of integration). However, there is no guarantee that a conformal mapping will in general exist for a given \( k^2(x, y) \). One case where this does apply is for integer powers of \( z \). The form of the Jacobian may be written in general as

\[
J(X, Y; x, y) = \left| \frac{df(z)}{dz} \right|^2
\]  

and when the mapping is based on a simple integer power of \( z \),

\[
Z = f(z) = z^m
\]

the Jacobian takes on a particular form, using \( r^2 = x^2 + y^2 \),

\[
J(X, Y; x, y) = m^2 (x^2 + y^2)^{m-1} = m^2 r^{2m-2}
\]

This is consistent with the derivation developed by Shaw and Gipson\(^9\) for the equation

\[
\nabla^2 W(\vec{r}) + \beta W(\vec{r}) = - \delta(\vec{r} - \vec{r}_0)
\]  

when \( \beta \) was equal to \( \alpha_n^2 r^{2n-2} \). (The reference actually used \( - \alpha_n^2 r^{2n-2} \) which led to modified Bessel functions rather than the standard Bessel function form found here, but the derivation is the same). The resultant solution for \( W = G(\vec{r}, \vec{r}_0) \) is

\[
G(\vec{r}, \vec{r}_0) = (i/4) H_0^{(1)}(\alpha_n |r^n - r_0^n| / n)
\]

If \( \beta \) is \( \alpha_n^2 r^{2n-2} \) and is also \( k_0^2 m^2 r^{2m-2} \), then \( m = n \) and \( \alpha_n = n k_0 \) and, as expected,

\[
G(\vec{r}, \vec{r}_0) = (i/4) H_0^{(1)}(k_0 |r^n - r_0^n|) = (i/4) H_0^{(1)}(k_0 |R - R_0|)
\]

However, this "newer" form allows for a far wider range of heterogeneities, e.g.

\[
f(z) = a_1 z + a_2 z^2 + \ldots + a_n z^n
\]

with a Jacobian

\[
J = |a_1 + 2 a_2 z + \ldots + n a_n z^{n-1}|
\]

is feasible as a conformal mapping even though it is not a simple power of \( z \) as required in reference 9.
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