Boundary element analysis of interface cracks in dissimilar orthotropic materials using a path independent contour integral
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Abstract
This paper presents a boundary element method for analysis of interface cracks in dissimilar materials. A path independent contour integral is used to obtain the complex stress intensity factors. For mixed-mode problems the crack opening displacements are used to decouple the stress intensity factors.

The proposed technique is applied to interface cracks in both isotropic and orthotropic dissimilar materials. Several examples are presented to demonstrate the accuracy of the method.

1 Introduction
The problem of cracks along the interface of orthotropic dissimilar materials has an important role in the structural analysis due to the extensive use of composites and reinforced materials in the modern industry. The strength of bonded joints between composite materials can be assessed through the knowledge of stress intensity factors of cracks located at the material's interface.

In the study of interface cracks, Williams[1], Erdogan[2] and Rice[3] describe the characteristic oscillating behaviour of stresses and displacements near the crack tip in dissimilar isotropic materials. Rice[3] used the Muskelishvili's complex function theory and defined the complex stress intensity factor associated with an elastic interface crack, ignoring the possible contact zone due to the oscillation of displacements. Some discussions are also presented in this work related to the validity of the complex stress intensity factor as a crack tip characterizing parameter.

Numerical results of displacements and stresses obtained with the boundary element method and the finite element method were used in several publications to calculate the complex stress intensity factor. For example, Lee and Choi[4], Yuuki and Cho[5], Tan and Gao[6] and Yuuki and Xu[7] used displacement values on the crack surfaces and traction values in points ahead of the crack tip. Xiao and Hui[8] and Miyazaki et al.[9] used the energy release rates in the crack. All the above works considered isotropic material. Cho et al.[10], Charalambides and Zhang[11] and Beuth[12] have presented solutions for interface cracks in orthotropic and anisotropic bimaterials.

In this work the complex stress intensity factors for interface cracks in orthotropic bimaterials are obtained using a path independent contour integral and the crack opening...
displacements. The study is first presented for the isotropic case to assess the accuracy of the results. The boundary element method is applied to obtain the numerical solution based on the fundamental solution for anisotropic medium.

2 Boundary Element Method for Orthotropic Material

Following Lekhnitskii[13] and using the notation of Sollero and Aliabadi[14], the governing equation for the anisotropic plane stress problem can be written in terms of a stress function $F(x,y)$ as:

$$S_{22} \frac{\partial^4 F}{\partial x^4} - 2S_{26} \frac{\partial^4 F}{\partial x^3 \partial y} + (2S_{12} + S_{66}) \frac{\partial^4 F}{\partial x^2 \partial y^2} - 2S_{16} \frac{\partial^4 F}{\partial x \partial y^3} + S_{11} \frac{\partial^4 F}{\partial y^4} = 0$$

(1)

where $S_{ij}$ are compliance coefficients relating stresses and strains. For orthotropic problems these coefficients are given by:

$$S_{11} = \frac{1}{E_1}, \quad S_{12} = S_{21} = -\frac{\nu_{12}}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{66} = \frac{1}{G_{12}}, \quad S_{16} = S_{26} = 0$$

(2)

where $E_k$ are the elastic modulus in the principal directions, $G_{12}$ is the shear modulus and $\nu_{12}$ is the Poisson's ratio.

Using the characteristic complex planes:

$$z_k = x + \mu_k y$$

(3)

where $\mu_k$, $k = 1, 2$ are the complex roots of the characteristic equation:

$$S_{22} + (2S_{12} + S_{66})\mu^2 + S_{11}\mu^4 = 0$$

(4)

the integral equation of the boundary element method without body forces is as follows:

$$c_{ij}(z^{'})u_j(z^{'}) + \int_\Gamma p_{ij}(z^{'},z)u_j(z) d\Gamma(z) = \int_\Gamma u_{ij}(z^{'},z)p_j(z) d\Gamma(z)$$

(5)

In the equation (5) $z'$ is the source point and $z$ the field point, $u_j(z)$ and $p_j(z)$ are the displacements and tractions in the boundary and $c_{ij}(z')$ is a coefficient that depends on the geometry of the boundary in the point $z'$. The kernels $u_{ij}(z^{'},z)$ and $p_{ij}(z^{'},z)$ correspond to the fundamental solution and have the following expression for orthotropic problems:

$$u_{ij}(z^{'},z) = 2 \text{Re}[p_{j1}A_{11} \ln(z_1 - z_1^{'}) + p_{j2}A_{12} \ln(z_2 - z_2^{'})]$$

(6)

$$p_{ij}(z^{'},z) = 2 \text{Re} \left[ \frac{1}{(z_1 - z_1^{'})} q_{j1}(\mu_1 n_1 - n_2)A_{11} + \frac{1}{(z_2 - z_2^{'})} q_{j2}(\mu_2 n_1 - n_2)A_{12} \right]$$

(7)

where $n_k$ are the components of the normal outward unitary vector,

$$[p_{jk}] = \begin{bmatrix} S_{11}\mu_k^2 + S_{12} \\ S_{12}\mu_k + S_{22}/\mu_k \end{bmatrix}$$

(8)

$$[q_{jk}] = \begin{bmatrix} \mu_1 & \mu_2 \\ -1 & -1 \end{bmatrix}$$

(9)
and the complex constants $A_{ik}$ are the solution of the linear equation systems:

$$
\begin{bmatrix}
1 & -1 & 1 & -1 \\
\mu_1 & -\mu_1 & \mu_2 & -\mu_2 \\
p_{11} & -p_{11} & p_{12} & -p_{12} \\
p_{21} & -p_{21} & p_{22} & -p_{22}
\end{bmatrix}
\begin{bmatrix}
A_{i1} \\
A_{i1} \\
A_{i2} \\
A_{i2}
\end{bmatrix}
= 
\begin{bmatrix}
\delta_{i2}/(2\pi i) \\
-\delta_{i1}/(2\pi i) \\
0 \\
0
\end{bmatrix}
$$

(10)

### 3 Stress Intensity Factors for Interface Cracks

The problem of interface cracks for dissimilar materials presents an oscillatory behaviour for stress and displacement fields in the vicinity of the crack tip, regardless of the material (isotropic or orthotropic). The oscillation in the displacement field results in an overlapping of crack surfaces that is taken in account in some works as a contact zone. However, since this zone is confined to a very small region near the crack tip, it can be ignored and the analysis is done considering that crack surfaces are traction free.

In the region near the crack tip, the displacements and stresses are parametrized through a complex stress intensity factor $K = K_I + iK_{II}$ where $K_I$ and $K_{II}$, as pointed out by Charalambides and Zhang[11], do not in general represent opening and sliding modes in bimaterial fractures. The modulus $|K| = \sqrt{K_I^2 + K_{II}^2}$ however is uniquely related to the energy release rate of the crack as in the homogeneous case.

For linear elastic bodies and the crack surfaces alined with the $X_1$ axes, the energy release rate of the crack ($G$) can be obtained through the $J$ contour integral (Sollero and Aliabadi[14]) defined as:

$$
G = J = \int_{\Gamma} W dx_2 - n_i \sigma_{ij} \frac{\partial u_j}{\partial x_1} ds
$$

(11)

where $W$ is the strain energy density, $\Gamma$ is an arbitrary contour starting and finishing in the crack surfaces, enclosing the crack tip, $n_i$ is the outward unit normal to the contour $\Gamma$ and $ds$ is an infinitesimal element of contour arc length. It can be shown that this integral is path independent in the absense of body forces and if the crack surfaces are traction free. In this work a circular contour $\Gamma$ as shown in Figure 1 is used.

In order to obtain the phase angle $\Psi = \arctan(K_{II}/K_I)$, the Crack Opening Displacement method is used for both isotropic and orthotropic materials.
3.1 Interface cracks in isotropic bimaterial

For a crack between two isotropic materials the stresses along $\theta = 0$ is given by:

$$\sigma_{22} + i\sigma_{12} = (K_I + iK_{II})(2\pi r)^{-1/2}(\frac{r}{l})^{i\varepsilon}$$  (12)

where $l$ is an arbitrary length to normalize the distance $r$ in the logarithmic term and usually taken as the total crack length. $\varepsilon$ is a bimaterial constant with the expression:

$$\varepsilon = \frac{1}{2\pi} \ln \left[ \frac{k_1 + \frac{1}{\mu_1}}{k_2 + \frac{1}{\mu_2}} \right]$$  (13)

where $k_j$ is the shear modulus and $k_j = (3 - \nu_j)/(1 + \nu_j)$ for plane stress or $k_j = 3 - 4\nu_j$ for plane strain ($\nu_j$ is the Poisson's ratio). The free subscript stands for each material. The jump in the displacements along the crack surfaces ($\theta = -\pi$ and $\theta = \pi$) is also given in a complex form:

$$\delta_2 + i\delta_1 = \frac{K_I + iK_{II}}{2(1 + 2i\varepsilon) \cosh(\varepsilon\pi)} \left[ \frac{k_1 + \frac{1}{\mu_1}}{k_2 + \frac{1}{\mu_2}} \right] \frac{r}{2\pi}^{1/2} \left( \frac{r}{l} \right)^{i\varepsilon}$$  (14)

Using equation (14) is possible to obtain the relationship $K_{II}/K_I$ by means of extrapolating the numerical results of displacements in points apart from the crack tip (around 10% of the crack length):

$$\frac{K_{II}}{K_I} = \lim_{r \to 0} \frac{1 - (\delta_2/\delta_1)A}{(\delta_2/\delta_1)A + A}$$  (15)

From equation (14) the modulus of the complex stress intensity factor $K$ can also be obtained:

$$|K| = \lim_{r \to 0} \left[ \frac{\sqrt{\delta_2^2 + \delta_1^2}}{\sqrt{2\pi r}} \right] \sqrt{\frac{1}{4\pi\sqrt{1 + \varepsilon^2} \cosh(\varepsilon\pi)} \left( \frac{k_1 + \frac{1}{\mu_1}}{k_2 + \frac{1}{\mu_2}} \right)}$$  (16)

Another option to obtain the modulus $|K|$ is through the energy release rate. As it will be shown, the use of this option results in more accurate numerical approximations for the stress intensity factor. The energy release rate, that is calculated in this work with the path independent contour integral, is related with the modulus $|K|$ as follows:

$$|K| = 4\cosh(\varepsilon\pi) \left[ J / \left( \frac{k_1 + \frac{1}{\mu_1}}{k_2 + \frac{1}{\mu_2}} \right) \right]^{1/2}$$  (17)

3.2 Interface cracks in orthotropic bimaterial

Similar expressions can be obtained for the case of orthotropic material. The stresses along $\theta = 0$ and displacement jumps along the crack surfaces are written as:

$$\sqrt{\frac{H_{22}}{H_{11}}} \sigma_{22} + i\sigma_{12} = (K_I + iK_{II})(2\pi r)^{-1/2}(\frac{r}{l})^{i\varepsilon}$$  (18)

$$\sqrt{\frac{H_{11}}{H_{22}}} \delta_2 + i\delta_1 = \frac{2H_{11}(K_I + iK_{II})}{(1 + 2i\varepsilon) \cosh(\varepsilon\pi)} \frac{(r)^{1/2}}{2\pi} \left( \frac{r}{l} \right)^{i\varepsilon}$$  (19)

with

$$\varepsilon = \frac{1}{2\pi} \ln \left[ \frac{1 - \beta}{1 + \beta} \right]$$  (20)

$$\beta = \left( \sqrt{S_{11}S_{22}} + S_{12} \right)_{\text{Mat2}} - \left( \sqrt{S_{11}S_{22}} + S_{12} \right)_{\text{Mat1}} / \sqrt{H_{11}H_{22}}$$  (21)
and

$$H_{11} = [2n\lambda^{1/4}\sqrt{S_{11}S_{22}}]_{\text{Mat}1} + [2n\lambda^{1/4}\sqrt{S_{11}S_{22}}]_{\text{Mat}2}$$

$$H_{22} = [2n\lambda^{-1/4}\sqrt{S_{11}S_{22}}]_{\text{Mat}1} + [2n\lambda^{-1/4}\sqrt{S_{11}S_{22}}]_{\text{Mat}2}$$

and

$$\lambda = \frac{S_{11}}{S_{22}}, \quad n = \sqrt{\frac{1}{2}(1 + \rho)} , \quad \rho = \frac{2S_{12} + S_{66}}{2\sqrt{S_{11}S_{22}}}$$

With the equation (19) of relative displacements the modulus $|K|$ and the phase angle $\Psi$ are given by:

$$|K| = \lim_{r \to 0} \left[ \frac{\sqrt{H_{11}^2 + H_{22}^2}}{\sqrt{2\pi r}} \right] \left[ \frac{H_{11}}{\pi \sqrt{1 + 4e^2 \cosh(\varepsilon)}} \right]$$

$$\Psi = \arctan\left( \frac{K_{II}}{K_I} \right) = \arctan\left( \delta_1 / \sqrt{H_{11}^2 / H_{22}^2} \right) - \varepsilon \ln(T) + \arctan(2\varepsilon)$$

The modulus $|K|$ as function of the energy release rate is as follows:

$$|K| = 2\cosh(\varepsilon) |J / H_{11}|^{1/2}$$

4 Numerical Results

Numerical results are now presented for classical interface crack problems and they are compared with the ones found in the literature. The first example considers isotropic material and is used to demonstrate the applicability of the technique. Then the application for orthotropic material is evaluated. The standard 3-node parabolic boundary element is used without special considerations at the crack tip and the different materials are considered with the subregion technique.

4.1 Interface Cracks in Isotropic Dissimilar Materials

The first example considers interface cracks in finite plates under a uniform tension. Figure 2 represents the geometry of the problem for the case of the single edge crack and the central crack. In the latter case, due to the symmetry, only half of the configuration is modelled. The boundary of each subregion was divided into 50 elements for $a/w = 0.3$ and 53 elements for $a/w = 0.4$, and a fine mesh was applied in the area close to the crack tip.

The results for a non-dimensional stress intensity factor $F = |K| / \sigma \sqrt{\pi a}$ and $K_{II}/K_I$, against various values of $E_1/E_2$, are shown in Table 1, for the single edge crack, and Table 2 for the central crack, and are compared with values given by Yuuki and Cho[5].

It can be seen that the Crack Opening Displacement (COD) method gives good results for the rate $K_{II}/K_I$ but not for the approximations of the modulus $|K|$, which are in best agreement with the values obtained via the J integral.

4.2 Interface Cracks in Orthotropic Dissimilar Materials

The next example is a central interface crack in a finite orthotropic plate with the same geometry and discretization shown in the Figure 2. In this case the material 1 is orthotropic with constant ratio $E_2/E_1 = 0.5$ and $\nu_{12} = 0.3$. The material defining parameter is given by the relation of its physical properties as:
Figure 2: (a) Single Edge Crack (b) Central Crack

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$a/w = 0.3$</th>
<th>$a/w = 0.4$</th>
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<tr>
<td></td>
<td>Ref.</td>
<td>J Int.</td>
</tr>
<tr>
<td>1</td>
<td>$F$</td>
<td>1.66</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
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<tr>
<td>2</td>
<td>$F$</td>
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<td>$F$</td>
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<td></td>
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<tr>
<td>10</td>
<td>$F$</td>
<td>1.693</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
<td>-0.242</td>
</tr>
<tr>
<td>100 (Ref.[9])</td>
<td>$F$</td>
<td>1.709</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
<td>-0.295</td>
</tr>
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Table 1: Stress Intensity Factors for a Single Edge Crack

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>$a/w = 0.3$</th>
<th>$a/w = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ref.</td>
<td>J Int.</td>
</tr>
<tr>
<td>1</td>
<td>$F$</td>
<td>1.058</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>$F$</td>
<td>1.055</td>
</tr>
<tr>
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<td>$K_{II}/K_I$</td>
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</tr>
<tr>
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<td>$F$</td>
<td>1.046</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
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</tr>
<tr>
<td>10</td>
<td>$F$</td>
<td>1.032</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
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</tr>
<tr>
<td>100</td>
<td>$F$</td>
<td>1.014</td>
</tr>
<tr>
<td></td>
<td>$K_{II}/K_I$</td>
<td>-0.202</td>
</tr>
</tbody>
</table>

Table 2: Stress Intensity Factors for a Central Crack
Table 3: Central Crack in a Finite Orthotropic Plate

The material 2 is also orthotropic and has a rate $E_2/E_1$ variable, keeping the Poisson’s ratio $\nu_{12}$ fixed at 0.3. Table 3 shows the results for various values of $E_2/E_1$ and $\varrho$ compared with the ones presented by Cho et al.[10]. The values of $F$ where obtained with the J integral and the relationship $K_{II}/K_I$ with the COD method. The only discrepancy observed was in the value of $K_{II}/K_I$ for $\varrho=3.32$ and $(E_2/E_1)_{Mat2}=0.1$ but, looking at the solution behaviour in the other cases, it seems that there is a printing mistake in Cho et al.[10] results.

5 Conclusions

In this work a combination of the Path Independent Contour Integral and the Crack Opening Displacement methods was used to obtain accurate complex stress intensity factors for interface cracks between orthotropic dissimilar materials. The Boundary Element Method was applied to obtain the numerical results using the standard 3-node isoparametric parabolic element. The results obtained are in very good agreement with those presented in the literature. As in other numerical procedures, suitable fine meshes are required in the crack tip vicinity but, as one of the main advantages of the proposed technique, there is no necessity to use special elements and further considerations at the crack tip.

References

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