The application of J-integrals in linear elastic problems in plate bending
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Abstract

The application of the Boundary Element Method for the determination of the stress intensity factors in cracked plates is discussed in this paper. The J integral method has been used within the Dual Boundary Element Method for the calculation of the stress intensity factors. A number of case studies have been used to illustrate the effectiveness of the technique for the determination of $K_I$, $K_{II}$ and $K_{III}$ stress intensity factors in cracked plates. The results from the case studies are compared with results from alternative approaches and good agreement has been achieved.

Introduction

The determination of the stress distribution in the vicinity of a crack, particularly in safety critical situations, is of vital importance to engineers, as the presence of a crack in a component can considerably reduce the strength of the component. Theoretical work on fracture mechanics has been well documented and predicts a singular stress condition at the crack tip. The prediction of the stress distribution in the region of crack is governed by the stress intensity factor at the crack tip. Due the importance of fracture analysis in assessing the structural integrity of components, there is a considerable amount of published data enabling engineers to determine the stress intensity factors for a wide range of component geometries, loading conditions and crack configurations. Although extensive, this handbook data is not exhaustive. Engineers are therefore constantly seeking additional approaches for the determination of stress intensity factors as components, loading
conditions and crack configurations become more complex and the structural integrity of components features more prominently at the design stage. The application of numerical techniques like the Finite Element Method (FEM) and the Boundary Element Method (BEM) have therefore become important for the determination of stress intensity factors in situations, where existing published hand book has been shown to be inadequate. As the BEM only requires the boundary of the problem to be modelled it has many advantages over the FEM, particularly in the field of linear elastic fracture mechanics. The BEM does however pose one major problem if the displacement boundary integral equations are used to model all boundaries, including the crack faces. In these circumstances, the equations modelling the crack faces lead to a system of singular algebraic equations. This problem can be resolved by using the Dual Boundary Element Method (DBEM), in which one face of the crack is modelled using the displacement boundary element equation and the other face of the crack is modelled using the traction boundary element equation. The remainder of the boundary of the component is modelled using the displacement boundary element equation. Two and three dimensional stress problems, with cracks have been analysed successfully using the DBEM\(^4,5\). There is however little published work relating to the application of the DBEM for the determination of stress intensity factors in plate bending problems. The work discussed in this paper therefore concerns the application of the DBEM for the fracture analysis of plate bending problems in LEFM, with the J integral technique being used to determine the stress intensity factors. Three case studies for a range of plate geometries, boundary conditions, loading conditions and crack configurations are considered in the paper. The J integral results are compared with analytical and finite element results and with results obtained using different approaches within the DBEM to determine stress intensity factors. In all cases the J integral results are in good agreement with the results obtained from alternative approaches.

**The Boundary Element Method for Plate Bending**

The plate bending boundary element equations, which are used in this paper relate to Reissner's\(^6\) plate bending equations and are given by Karam and Telles\(^7\). The displacements at an internal point on the plate are obtained from the following equation

\[
-u_i(\xi) + \int_{\Gamma} T_{ij}^*(\xi, x) u_j(x) d\Gamma(x) = \int_{\Gamma} U_{ij}^*(\xi, x) t_j(x) d\Gamma(x) + \int_{\Omega} \left( U_{13}^*(\xi, x) - \frac{\nu}{(1 - \nu)\lambda^2} U_{i\alpha, \alpha}^*(\xi, x) \right) q(x) d\Gamma(x)
\] (1)
in which \( x \) is a point on the boundary of the plate and \( U^*_{ij}(\xi, x) \) and \( T^*_{ij}(\xi, x) \) are the fundamental solutions for displacement and traction respectively. When the distance, \( r \), between \( \xi \) and \( x \) is not zero, the integrals in equation (1) are regular. The integrals become singular, however, when \( \xi \) is moved to the boundary of the plate and \( \xi \) coincides with \( x \) and in these circumstances equation (1) becomes

\[
c_{ij}(\xi)u_j(\xi) + (\text{CPV}) \int \Gamma T^*_{ij}(\xi, x)u_j(x)d\Gamma(x) = \int \Gamma U^*_{ij}(\xi, x)t_j(x)d\Gamma(x) + \int_{\Omega} \left( U^*_{i3}(\xi, x) - \frac{\nu}{(1-\nu)\lambda^2} U^*_{i\alpha\alpha}(\xi, x) \right) q(x)d\Gamma(x)
\]  

(2)

where \((\text{CPV})\) denotes a Cauchy Principal Value Integral.

Stresses at internal points are obtained from integral equations for bending moment and shear force, which are derived from equations (1) and (2) as

\[
M_{\alpha\beta}(\xi) = \int \Gamma D^*_{\alpha\beta k}(\xi, x)t_k(x)d\Gamma(x) - \int \Gamma S^*_{\alpha\beta k}(\xi, x)u_k(x)d\Gamma(x) + q\int_{\Omega} W^*_{\alpha\beta}(\xi, x)d\Gamma(x) + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta}
\]  

(3)

\[
Q_{\beta}(\xi) = \int \Gamma D^*_{3\beta k}(\xi, x)t_k(x)d\Gamma(x) - \int \Gamma S^*_{3\beta k}(\xi, x)u_k(x)d\Gamma(x) + q\int_{\Omega} W^*_{3\beta}(\xi, x)d\Gamma(x)
\]  

(4)

Equations (3) and (4) also become singular when \( \xi \) is moved to the boundary and \( \xi \) and \( x \) coincide. In these circumstances, provided the boundary is smooth, these equations become

\[
\frac{1}{2} M_{\alpha\beta}(\xi) = (\text{CPV}) \int \Gamma D^*_{\alpha\beta k}(\xi, x)t_k(x)d\Gamma(x) - (\text{HPV}) \int \Gamma S^*_{\alpha\beta k}(\xi, x)u_k(x)d\Gamma(x) + \left[ (\text{CPV}) \int \Gamma W^*_{\alpha\beta}(\xi, x)d\Gamma(x) + \frac{\nu}{(1-\nu)\lambda^2} q \delta_{\alpha\beta} \right] q
\]  

(5)
\[ \frac{1}{2} Q_\beta(\xi) = (\text{CPV}) \int_{\Gamma} D^*_{3\beta k}(\xi, x) t_k(x) d\Gamma(x) - (\text{HPV}) \int_{\Gamma} S^*_{3\beta k}(\xi, x) u_k(x) d\Gamma(x) + q \int_{\Gamma} W^*_{3\beta}(\xi, x) d\Gamma(x) \] (6)

in which (HPV)\(\int\) represents a Hadamard Principal Value Integral.

The components of traction, \( t_i \), are obtained from

\[ t_\alpha = M_{\alpha\beta} n_\beta \text{ and } t_3 = Q_\alpha n_\alpha \] (7)

in which \( n \) represents the outward normal.

Equations (5), (6) and (7) can therefore be used to obtain the traction components on a smooth boundary as

\[ \frac{1}{2} t_\alpha(\xi) = n_\beta(\xi)(\text{CPV}) \int_{\Gamma} D^*_{\alpha\beta k}(\xi, x) t_k(x) d\Gamma(x) - n_\beta(\xi)(\text{HPV}) \int_{\Gamma} S^*_{\alpha\beta k}(\xi, x) u_k(x) d\Gamma(x) \]

\[ + \left[ n_\beta(\xi)(\text{CPV}) \int_{\Gamma} W^*_{\alpha\beta}(\xi, x) d\Gamma(x) + \frac{v}{(1-v)\lambda^2} n_\alpha(\xi) \right] q \] (8)

\[ \frac{1}{2} t_3(\xi) = n_\beta(\xi)(\text{CPV}) \int_{\Gamma} D^*_{3\beta k}(\xi, x) t_k(x) d\Gamma(x) - n_\beta(\xi)(\text{HPV}) \int_{\Gamma} S^*_{3\beta k}(\xi, x) u_k(x) d\Gamma(x) + q n_\beta(\xi) \int_{\Gamma} W^*_{3\beta}(\xi, x) d\Gamma(x) \] (9)

Equations (2), (8) and (9) are the integral equations which constitute the Dual Boundary Element Method using Reissner’s plate bending equations and are used in the analysis of the case studies discussed in this paper.

**Case Studies**

Three case studies are presented in this paper to illustrate the application of the J-integral method within the DBEM.
Case Study 1 (Fig. 1).

A cantilevered rectangular plate acted on by a uniformly distributed moment on the free edge opposite the built-in edge. The plate has an interior circular hole and a single interior crack, parallel to the built-in edge, as shown in figure 1. Each external boundary of the plate was modelled using four continuous quadratic elements. The circumference of the hole was modelled using twelve continuous quadratic elements and each crack face was modelled using eight discontinuous quadratic elements. Each crack tip was modelled used discontinuous quarter point elements.

Case Study 2 (Fig. 2)

This case study comprises a cantilevered rectangular plate acted on by a uniformly distributed moment on the free edge, parallel to the built-in edge. The plate has a circular interior hole and an interior crack, parallel to the fixed edge, emanating from the hole, as shown in figure 2. Four quadratic continuous boundary elements were used to model each external boundary of the plate. The hole was modelled using continuous quadratic elements, both faces of the crack were modelled using discontinuous quadratic elements, with the exception of the junction between the crack and the hole, where both the hole and the crack were modelled using semi-continuous quadratic elements. Twelve elements were used to model the hole and each face of the crack was modelled using four elements. Quarter point elements were used at the crack tip.

Case Study 3 (Fig. 3)

This case study comprises a rectangular plate with one short boundary and half of each long boundary simply supported and the remainder of the boundary free. A uniform shearing force acts on the short free boundary of the plate. The plate has a single crack perpendicular to the boundary of the plate at the junction of the simply supported and the free portions of one of the long boundaries as shown in figure 3. The boundary element model for this case study consisted of thirty two quadratic elements. Each long external boundary was modelled using eight elements, each short external boundary was modelled using four elements and each crack face was modelled using four elements. Discontinuous elements were used to model the crack face, semi-continuous elements were used at the junction of the crack face and the external boundary and continuous elements were used to model the remainder of the boundary. Quarter point elements were used at the crack tip.
In all case studies one face of the crack was modelled using the displacement boundary integral equation as given by equation (2) and the other crack face was modelled using the traction boundary integral equations as expressed by equations (8) and (9).

Results

In all case studies considered in this paper the plates were analysed using the DBEM and the stress intensity factors at the crack tip were determined for LEFM using the J-integral method and the displacement extrapolation method along the face of the crack. For case studies one and two, the $K_I$ stress intensity factor has been calculated for a range of crack lengths and for two plate thicknesses. The $K_p$, $K_{II}$ and $K_{III}$ stress intensity factors have been determined for case study three for a range of crack lengths and for one plate thicknesses.

For case study one the distance from the centre of the crack to the circumference of the hole equals the radius of the hole. The stress intensity factors were calculated for a range of values of $a/d$ from 0.1 to 0.9, where $a$ is the semi-crack length and $d$ is the distance from the centre of the crack to the circumference of the hole as shown in figure 1. Values of the variation of stress intensity factor are given in figure 4 for values of $t/d\sqrt{10}$ and 0.1 and 1.0, where $t$ is the thickness of the plate. Finite element results, using the J-integral approach to calculate the stress intensity factors, and Murakami's analytical results are also shown on figure 4 for the range of crack lengths and plate thicknesses, for which DBEM results have been obtained.

Results for case study 2 are shown in figure 5. In this case the variation of stress intensity factor with the ratio of crack length, $a$, to radius of hole, $r$, has been calculated for a range of values of $a/r$ from 0.1 to 2.0. Values of the variation of stress intensity factor are presented for values of $t/W$ of 0.1 and 1.0 where $t$ is the plate thickness and $W$ is its width as shown in figure 2.

The results for case study 3 are shown in figure 6. The stress intensity factors have been calculated for a range of values of $a/W$ from 0.1 to 0.9 where $a$ is the crack length and $w$ is the width of the plate. The stress intensity factors have been obtained for a value of $t/W$ of 0.5 where $t$ is the plate thickness and $w$ is its width as shown in figure 6.

In case studies one and two the results for the variation of the $K_I$ stress intensity have been obtained from the following expression as given by Murakami:

$$K_I = \frac{6M\sqrt{a}}{t^2} F_I$$
In case study three the $K_I$, $K_{II}$ and $K_{III}$ stress intensity factors were obtained from the following expressions as given by Hasabe

\[
K_I = \frac{6PCW}{t^2\sqrt{b}} F_I \tag{13}
\]
\[
K_{II} = \frac{6PCW}{t^2\sqrt{b}} F_{II} \tag{14}
\]
\[
K_{III} = \frac{6PW}{t^2\sqrt{b}} F_{III} \tag{15}
\]

The factors $F_I$, $F_{II}$ and $F_{III}$ in equations (12) to (15) relate the ratio of crack length, $a$, to plate width, $W$. In equations (13) and (14) $C$ is the length of the free portion of the long boundary in case study three.

**Discussion of Results**

The results presented in figures 4, 5 and 6 for the plates shown in figures 1, 2 and 3 clearly indicate the effectiveness of the DBEM, using the J integral method to calculate the stress intensity factors in the LEFM analysis of plate bending problems. Plates with a number of loading conditions, boundary conditions and crack configurations have been analysed.

Figure 4 and 5 give the results for the $K_I$ stress intensity factors for case studies one and two (Figs. 1 and 2) for a range of crack lengths and two plate thicknesses. The J integral results are in close agreement with the DBEM displacement extrapolation results and the finite element, J-integral results. There is however a difference of around ten per cent between the boundary element results and Murakami's\textsuperscript{5} analytical results for case study one. Results are presented in figure 6 for the $K_{III}$, $K_{II}$ and $K_{III}$ stress intensity factors for case study three (Fig. 3) for a plate with a range of crack lengths. The J-integral DBEM results in this case study are also in close agreement with the DBEM displacement extrapolation results.

In case study one, both sets of DBEM results have been achieved with considerably fewer elements and degrees of freedom than the finite element results. For example in case study one the boundary element model comprised 44 elements whereas the finite element model comprised 1500 elements.
Conclusions

The results of this study confirm that the DBEM can be used successfully for the determination of stress intensity factors in plate bending problems, when LEFM conditions apply. Both the J integral method and the displacement extrapolation method have been used to obtain $K_I$, $K_{II}$ and $K_{III}$ stress intensity factors for a range of plate bending problems. The boundary element results for the first case study have been compared with finite element results and the DBEM results are in close agreement with the FEM results. The DBEM has also been shown to have a clear advantage over the FEM, as results of comparable accuracy have been achieved from the DBEM, with considerably fewer elements and degrees of freedom, compared to the FEM. The work discussed in this paper clearly indicates that the DBEM can be used successfully for the fracture analysis of cracked plates.

References


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![Fig. 1 Case Study One](image1)

![Fig. 2 Case Study Two](image2)

![Fig. 3 Case Study Three](image3)
Fig. 4 Stress Intensity Factors for Case Study One

Fig. 5 Stress Intensity Factors for Case Study Two

Fig. 6 Stress Intensity Factors for Case Study Three