A hypersingular-boundary integral equation method for cracks in an anisotropic elastic half-space with a partly irregular boundary

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Abstract

The two-dimensional problem of analysing the interaction between planar cracks and the boundary of an anisotropic elastic half-space is formulated in terms of a system of hypersingular-boundary integral equations. The boundary of the half-space is partly irregular. For a specific example, the integral equations are solved numerically.

1 Introduction

The analysis of stress around cracks in an elastic half-space bounded by a flat surface (plane) has been carried out by many researchers, e.g. Nied [7], Sweeney [8] and Uzel *et al.* [9]. The present paper is concerned with the two-dimensional problem of arbitarily-located planar cracks in an anisotropic elastic half-space having an exterior boundary which is partly curved (irregular). The half-space is subject to a uniform far field stress which is such that the cracks open up. The exterior boundary of the half-space is free of traction.

The displacement and stress fields are suitably superimposed in such a way that the far field conditions are satisfied and the displacement is given by boundary integral equations taken over only the cracks and the exterior boundary of the half-space. If the irregular part of the exterior boundary is a depression ("valley"), it is possible to avoid integrating over the regular parts by using an appropriate Green's function. The boundary integral equations are employed to take care of the conditions on the exterior boundary of the half-space. The conditions on the cracks are expressed using a differentiated form of the integral equations, as in Ang [1], Ang and Noone [2] and Chen and Chen [3]. This approach leads to a system of

boundary integral equations coupled with hypersingular integral equations to be solved. For a specific example, the hypersingular-boundary integral equations are solved numerically.

2 Statement of the problem

Consider an anisotropic elastic half-space which is in equilibrium and contains N arbitrarily-located planar cracks in its interior. More specifically, with reference to a Cartesian coordinate frame $0x_1x_2x_3$, the geometries of the cracks and the exterior boundary of the half-space are independent of the coordinate x_3 and the half-space occupies the region $x_2 < f(x_1)$. Here $f(x_1)$ is a given function which is such that $f(x_1)$ is continuous over $(-\infty, \infty)$, $f(x_1) = 0$ for $|x_1| \geq a$ and $f(x_1)$ is differentiable over $(-a, a)$ (a is a given positive constant). Thus, $x_2 = f(x_1)$ for $-a < x_1 < a$ describes the curved part of the exterior boundary of the half-space. On the $0x_1x_2$ plane, the tips of the k-th crack are given by $(\alpha^{(k)}, \beta^{(k)})$ and $(\gamma^{(k)}, \delta^{(k)})$. The half-space is subject to a uniform *far-field* uniaxial stress given by $\sigma_{ij}^{(0)} = \delta_{i1}\delta_{j1}T$, where δ_{ij} is the Kronecker-delta and T is a given positive constant. The exterior boundary $x_2 = f(x_1)$ of the half-space and the cracks are traction-free. We are interested in analysing the interaction between the cracks and the partly irregular boundary of the half-space.

3 Integral equations

For the solution of the problem described above, let the displacement u_k and the stress σ_{ij} respectively be given by

$$u_k = u_k^{(0)} + u_k^{(1)} \quad \text{and} \quad \sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(1)} \tag{1}$$

where $u_k^{(0)}$ is the displacement which gives rise to the uniform stress field $\sigma_{ij}^{(0)} = \delta_{i1}\delta_{j1}T$. The displacement $u_k^{(1)}$ and the corresponding stress $\sigma_{ij}^{(1)}$, yet to be determined, are required to vanish in the far-field and are to be chosen to cancel out the tractions due to the uniform uniaxial stress on the cracks and the exterior boundary of the half-space.

Application of the boundary integral equations for plane anisotropic elasticity (given in Clements [4]) to the problem yields:

$$
\begin{aligned}
&\lambda(\xi_1,\xi_2)u_j^{(1)}(\xi_1,\xi_2)\\
=\ &\int_{-a}^{a}\left[u_i^{(1)}(x,f(x))\Gamma_{ij}(x,f(x);\xi_1,\xi_2;m_1(x),m_2(x))\right.\\
&\left.+\Phi_{1j}(x,f(x);\xi_1,\xi_2)Tm_1(x)\right]\sqrt{1+[f'(x)]^2}dx\\
&+\int_a^{\infty}u_i^{(1)}(x,0)\Gamma_{ij}(x,0;\xi_1,\xi_2;0,1)dx\\
&+\int_{-\infty}^{-a}u_i^{(1)}(x,0)\Gamma_{ij}(x,0;\xi_1,\xi_2;0,1)dx\\
&+\frac{1}{2}\int_{-1}^{1}\sum_{n=1}^{N}\Delta u_i^{(1)}(X_1^{(n)}(t),X_2^{(n)}(t))\\
&\qquad\times\Gamma_{ij}(X_1^{(n)}(t),X_2^{(n)}(t);\xi_1,\xi_2;N_1^{(n)},N_2^{(n)})L^{(n)}dt, \qquad (2)
\end{aligned}
$$

where $\lambda(\xi_1,\xi_2)=0$ if $\xi_2>f(\xi_1)$, $\lambda(\xi_1,\xi_2)=1/2$ if $\xi_2=f(\xi_1)$ but $\xi_1\neq\pm a$ and $\lambda(\xi_1,\xi_2)=1$ if $\xi_2<f(\xi_1)$, $m_1(x)=-f'(x)/\sqrt{1+[f'(x)]^2}$, $m_2(x)=1/\sqrt{1+[f'(x)]^2}$, $L^{(n)}=\sqrt{(\alpha^{(n)}-\gamma^{(n)})^2+(\beta^{(n)}-\delta^{(n)})^2}$, $N_1^{(n)}=-\left(\beta^{(n)}-\delta^{(n)}\right)/L^{(n)}$, $N_2^{(n)}=\left(\alpha^{(n)}-\gamma^{(n)}\right)/L^{(n)}$, $2X_1^{(n)}(t)=\alpha^{(n)}+\gamma^{(n)}+\left(\gamma^{(n)}-\alpha^{(n)}\right)t$, $2X_2(t)=\beta^{(n)}+\delta^{(n)}+\left(\delta^{(n)}-\beta^{(n)}\right)t$, $\Delta u_i^{(1)}(X_1^{(n)}(t),X_2^{(n)}(t))$ give the jump in the displacements across opposite faces of the n-th crack and

$$
\begin{aligned}
\Phi_{kj}(x_1,x_2;\xi_1,\xi_2) &= \frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{\alpha=1}^{2}A_{k\alpha}N_{\alpha p}\ln(z_\alpha-c_\alpha)\right\}d_{pj}\\
&\quad+\Phi^*_{kj}(x_1,x_2;\xi_1,\xi_2),\\
\Gamma_{ij}(x_1,x_2;\xi_1,\xi_2;n_1,n_2) &= \frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{\alpha=1}^{2}L_{ip\alpha}N_{\alpha r}(z_\alpha-c_\alpha)^{-1}\right\}d_{rj}n_p\\
&\quad+\Gamma^*_{ij}(x_1,x_2;\xi_1,\xi_2;n_1,n_2), \qquad (3)
\end{aligned}
$$

where $z_\alpha=x_1+\tau_\alpha x_2$, $c_\alpha=\xi_1+\tau_\alpha\xi_2$ and Re denotes the real part of a complex number. The constants τ_α, $A_{k\alpha}$, $N_{\alpha p}$, d_{pj} and $L_{ip\alpha}$ are related to the elastic moduli of the material occupying the half-space and are defined in Clements [4]. Throughout the paper, unless stated otherwise, the convention of summing over a repeated index is adopted only for latin subscripts running from 1 to 2. The functions Φ^*_{kj} and Γ^*_{ij} can always be chosen to be given by $\Phi^*_{kj}=0$ and $\Gamma^*_{ij}=0$. For certain special cases of the problem, however, it may be advantageous to choose these functions differently (see, e.g. Clements and Haselgroves [5]).

Using (2) and letting (ξ_1,ξ_2) approach the p-th crack, we find that the conditions that the cracks are traction-free give rise to:

$$
\begin{aligned}
&\Big\{\int_{-a}^{a}\Big[u_i^{(1)}(x,f(x))\Gamma_{ij,s}(x,f(x);X_1^{(n)}(\xi),X_2^{(n)}(\xi);m_1(x),m_2(x))\\
&+\Phi_{1j,s}(x,f(x);X_1^{(n)}(\xi),X_2^{(n)}(\xi))Tm_1(x)\Big]\sqrt{1+[f'(x)]^2}dx\\
&+\int_{a}^{\infty}u_i^{(1)}(x,0)\Gamma_{ij,s}(x,0;X_1^{(n)}(\xi),X_2^{(n)}(\xi);0,1)dx\\
&+\int_{-\infty}^{-a}u_i^{(1)}(x,0)\Gamma_{ij,s}(x,0;X_1^{(n)}(\xi),X_2^{(n)}(\xi);0,1)dx\\
&+\frac{1}{2}\mathcal{H}\int_{-1}^{1}\sum_{n=1}^{N}\Delta u_i^{(1)}(X_1^{(n)}(t),X_2^{(n)}(t))\\
&\times\Gamma_{ij,s}(X_1^{(n)}(t),X_2^{(n)}(t);X_1^{(n)}(\xi),X_2^{(n)}(\xi);N_1^{(n)},N_2^{(n)})L^{(n)}dt\Big\}c_{kqjs}N_q^{(p)}\\
=\;&-\delta_{k1}N_1^{(p)}T\quad\text{for}\;-1<\xi<1\;(p=1,2,\cdots,N),
\end{aligned}
\tag{4}
$$

where $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense, $\Phi_{kj,s}(x_1,x_2;\xi_1,\xi_2)$ and $\Gamma_{ij,s}(x_1,x_2;\xi_1,\xi_2;n_1,n_2)$ respectively denote the partial derivatives of $\Phi_{kj}(x_1,x_2;\xi_1,\xi_2)$ and $\Gamma_{ij}(x_1,x_2;\xi_1,\xi_2;n_1,n_2)$ with respect to ξ_s.

Equations (2) and (4) constitute a system of hypersingular-boundary integral equations which can be discretised simultaneously to solve approximately for the unknown functions $\Delta u_i^{(1)}(X_1^{(n)}(t),X_2^{(n)}(t))$ and the unknown displacement on the exterior boundary of the half-space. Once all these unknowns are determined, the displacement (and hence the stress) at any point (ξ_1,ξ_2) in the interior of the half-space can be readily computed using (2).

In the section below, we solve (2) and (4) numerically for a specific example of the problem. The numerical procedure employed to solve the equations is described in detail.

4 A specific example

For a specific example of the problem, we take the function $f(x_1)$ to be given by

$$
f(x_1)=\begin{cases}k(x_1^2-a^2)^2 & \text{if}\quad |x_1|<a\\ 0 & \text{if}\quad |x_1|\geq a\end{cases}
\tag{5}
$$

where k is a constant parameter.

The interior of the half-space contains only a single planar crack in the region $x_1=0$, $-(b+2\ell)<x_2<-b$, $-\infty<x_3<\infty$, where b and ℓ are real constants such that $b>-ka^4$ and $\ell>0$. We assume that the crack does not intersect with the exterior boundary of the half-space.

Firstly, let us consider the case where $k\leq 0$. The curved part of the exterior boundary of the half-space forms a "valley" if $k<0$. For $k\leq 0$, we

may take Φ^*_{kj} and Γ^*_{ij} in (3) to be given by

$$\begin{aligned}
&\Phi^*_{kj}(x_1,x_2;\xi_1,\xi_2)\\
&= -\frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{\alpha=1}^{2} A_{k\alpha}M_{\alpha q}\sum_{\beta=1}^{2}\overline{L}_{q2\beta}\overline{N}_{\beta r}\ln(z_\alpha-\overline{c}_\beta)\right\}d_{rj},\\
&\Gamma^*_{ij}(x_1,x_2;\xi_1,\xi_2;n_1,n_2)\\
&= -\frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{\alpha=1}^{2} L_{ip\alpha}M_{\alpha k}\sum_{\beta=1}^{2}\overline{L}_{k2\beta}\overline{N}_{\beta r}(z_\alpha-\overline{c}_\beta)^{-1}\right\}n_p d_{rj}, \qquad (6)
\end{aligned}$$

where $[M_{\alpha k}]$ is the inverse of $[L_{i2\alpha}]$. From (3) together with (6), it can be shown that $\Gamma_{ij}(x_1,0;\xi_1,\xi_2;0,1)=0$ for all x_1 and hence the integrals over $[a,\infty)$ and $(-\infty,-a]$ in (2) and (4) vanish.

Thus, for $k<0$, if we choose Φ^*_{kj} and Γ^*_{ij} to be given by (6), we find that (2) and (4) respectively become

$$\begin{aligned}
\frac{1}{2}V_j(\xi) &= \int_{-a}^{a}[V_i(x)\Gamma_{ij}(x,f(x);\xi,f(\xi);m_1(x),m_2(x))\\
&\quad+\Phi_{1j}(x,f(x);\xi,f(\xi))Tm_1(x)]\sqrt{1+[f'(x)]^2}dx\\
&+\ell\int_{-1}^{1}\Delta u_i(t)\Gamma_{ij}(0,Y(t);\xi,f(\xi);1,0)dt \quad \text{for } -a<\xi<a, \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\int_{-a}^{a}[V_i(x)\Gamma_{ij,s}(x,f(x);0,Y(\xi);m_1(x),m_2(x))\right.\\
&\left.\quad+\Phi_{1j,s}(x,f(x);0,Y(\xi))Tm_1(x)]\sqrt{1+[f'(x)]^2}dx\right\}c_{k1js}\\
&+\ell c_{k1js}\int_{-1}^{1}\Delta u_i(t)\Gamma^*_{ij,s}(0,Y(t);0,Y(\xi);1,0)dt\\
&+\frac{1}{2\pi}\,\mathrm{Re}\left\{\sum_{\alpha=1}^{2}\frac{Q_{ik\alpha}}{\tau_\alpha^2}\right\}\mathcal{H}\int_{-1}^{1}\frac{\Delta u_i(t)}{\ell}\cdot\frac{dt}{(t-\xi)^2}\\
&= -\delta_{k1}T \quad \text{for } -1<\xi<1, \qquad (8)
\end{aligned}$$

where $V_i(x)=u_i^{(1)}(x,f(x))$, $\Delta u_i(t)=\Delta u_i^{(1)}(0,Y(t))$, $Y(t)=-b-\ell+\ell t$ and $Q_{ik\alpha}=[c_{k1j1}+\tau_\alpha c_{k1j2}]\,L_{i1\alpha}N_{\alpha r}d_{rj}$.

We discretise the interval $[-a,a]$ into J equal length subintervals: $[a_0,a_1]$, $[a_1,a_2],\cdots,[a_{k-1},a_k],\cdots,[a_{J-2},a_{J-1}]$ and $[a_{J-1},a_J]$, where $a_k=-a+k(2a/J)$. We approximate $V_i(x)$ as constants given by $v_i^{(k)}$ over the subinterval $[a_{k-1},a_k]$.

Following Kaya and Erdogan [6], for the crack opening displacements, we make the approximation:

$$\Delta u_i(t)\simeq\sqrt{1-t^2}\sum_{m=1}^{M}\phi_i^{(m)}U_{m-1}(t), \qquad (9)$$

where $\phi_i^{(m)}$ are constants to be determined and $U_n(x)$ is the n-th order Chebyshev polynomial of the second kind.

With the approximations above, we find that (7) and (8) may be approximately superceded by (for $p = 1, 2, \cdots, J$)

$$\begin{aligned}
&\sum_{q=1}^{J} v_i^{(q)} \left\{ \frac{1}{2}\delta_{ij}\delta_{pq} \right. \\
&\left. -\sqrt{1+[f'(\hat{a}_q)]^2} \int_{a_{q-1}}^{a_q} \Gamma_{ij}(x, f(x); \hat{a}_p, f(\hat{a}_p); m_1(\hat{a}_q), m_2(\hat{a}_q))dx \right\} \\
&-\ell \sum_{m=1}^{M} \phi_i^{(m)} \int_{-1}^{1} \sqrt{1-t^2} U_{m-1}(t) \Gamma_{ij}(0, Y(t); \hat{a}_p, f(\hat{a}_p); 1, 0)dt \\
= &\sum_{q=1}^{J} T m_1(\hat{a}_q) \sqrt{1+[f'(\hat{a}_q)]^2} \int_{a_{q-1}}^{a_q} \Phi_{1j}(x, f(x); \hat{a}_p, f(\hat{a}_p))dx \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{q=1}^{J} v_i^{(q)} \sqrt{1+[f'(\hat{a}_q)]^2} \int_{a_{q-1}}^{a_q} c_{k1js} \Gamma_{ij,s}(x, f(x); 0, Y(\xi); m_1(\hat{a}_q), m_2(\hat{a}_q))dx \\
&+\ell \sum_{m=1}^{M} \phi_i^{(m)} \int_{-1}^{1} \sqrt{1-t^2} U_{m-1}(t) c_{k1js} \Gamma^*_{ij,s}(0, Y(t); 0, Y(\xi); 1, 0)dt \\
&-\frac{1}{2\ell} \mathrm{Re} \left\{ \sum_{\alpha=1}^{2} \frac{Q_{ik\alpha}}{\tau_\alpha^2} \right\} \sum_{m=1}^{M} \phi_i^{(m)} m U_{m-1}(\xi) \\
= &-\delta_{k1} T - \sum_{q=1}^{J} T m_1(\hat{a}_q) \sqrt{1+[f'(\hat{a}_q)]^2} \int_{a_{q-1}}^{a_q} c_{k1js} \Phi_{1j,s}(x, f(x); 0, Y(\xi))dx \\
&\text{for } -1 < \xi < 1, \qquad (11)
\end{aligned}$$

where $\hat{a}_q = (a_{q-1} + a_q)/2$.

The systems (10) and (11) contain a total of $(2J + 2)$ equations in $(2J + 2M)$ unknowns $v_i^{(q)}$ and $\phi_i^{(m)}$ $(i = 1, 2; q = 1, 2, \cdots, J; m = 1, 2, \cdots, M)$. To generate additional equations to form a system of $(2J + 2M)$ linear algebraic equations in the $(2J + 2M)$ unknowns, we choose ξ in (11) to be given, in turn, by

$$\xi = \xi_p \equiv \cos\left(\frac{[2p-1]\pi}{2M}\right) \quad \text{for } p = 1, 2, \cdots, M. \qquad (12)$$

The system of linear algebraic equations thus generated can be easily solved by using standard computer packages.

Once we have solved for $v_i^{(q)}$ and $\phi_i^{(m)}$ from the system of linear algebraic equations, the crack energy E defined by

$$E = -\frac{T}{2} \int_{-b-2\ell}^{-b} \Delta u_1^{(1)}(0, \xi)d\xi \qquad (13)$$

can be evaluated approximately using

$$E \simeq -\frac{\pi \ell T \phi_1^{(1)}}{4}. \tag{14}$$

If $b/\ell \to \infty$, i.e. if the interaction between the crack and the exterior boundary of the half-space is negligible, the crack energy is given by

$$E_0 = -\frac{\pi \ell^2 T^2}{2\chi} \quad (\chi = \mathrm{Re}\left\{\sum_{\alpha=1}^{3} \frac{Q_{11\alpha}}{\tau_\alpha^2}\right\}). \tag{15}$$

In other words, E_0 is the crack energy for the corresponding problem where the single planar crack lies in an anisotropic full-space.

We shall now compute the non-dimensionalised crack energy E/E_0 for a transversely-isotropic material whose elastic behaviour is governed by the equilibrium equations

$$\begin{aligned} C\frac{\partial^2 u_1}{\partial x_1^2} + L\frac{\partial^2 u_1}{\partial x_2^2} + (F+L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} &= 0, \\ A\frac{\partial^2 u_2}{\partial x_2^2} + L\frac{\partial^2 u_2}{\partial x_1^2} + (F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} &= 0, \end{aligned} \tag{16}$$

where A, N, F, C and L are elastic constants of the material. Constants such as τ_α, $A_{k\alpha}$, $N_{\alpha p}$, d_{pj} and $L_{ip\alpha}$, required in our computation, can be easily expressed in terms of A, N, F, C and L (see [4] or [5]). To be more specific, we shall use the elastic constants for titanium in our computation, i.e. we choose $A = 1.62$, $N = 0.92$, $F = 0.69$, $C = 1.81$ and $L = 0.467$ (in gram per centimeter per microsecond square).

Table 1: Crack energy E/E_0 vs b/ℓ for $k\ell^3 = -1$ and $a/\ell = 1$.

b/ℓ	1.200	1.400	1.600	1.800	2.000	3.000	4.000	8.000
E/E_0	1.402	1.251	1.179	1.137	1.109	1.047	1.026	1.006

For a fixed $k\ell^3 = -1$ and $a/\ell = 1$, the numerical values of E/E_0 against various values of $b/\ell > 1$ are calculated and given in Table 1. It is clear that the crack energy decreases with increasing b/ℓ, i.e. the crack becomes more stable if we move it away from the exterior boundary of the half-space. Furthermore, as expected, we observe that $E/E_0 \to 1^+$ as $b/\ell \to \infty$.

Table 2: Crack energy E/E_0 vs $k\ell^3$ for $a/\ell = b/\ell = 1$.

$k\ell^3$	0.000	−0.200	−0.400	−0.600	−0.800	−0.900
E/E_0	1.069	1.081	1.115	1.190	1.360	1.555

For a fixed $a/\ell = b/\ell = 1$, the numerical values of E/E_0 against various values of $k\ell^3$ ($-1 < k\ell^3 \leq 0$) are computed and tabulated in Table 2. Notice

that the crack energy increases as $k\ell^3$ gets closer and closer to -1 (from above).

Typically, the numerical values in Tables 1 and 2 are obtained by using $J = 20$ and $M = 10$. When the calculations are repeated using $J = 40$ and $M = 20$, convergence of the numerical values to at least 3 significant figures is observed.

For the case where $k > 0$, i.e. the case where the curved part of the exterior boundary of the half-space forms a "ridge," it is not permissible to choose Φ^*_{kj} and Γ^*_{ij} as given by (6). This is because Φ^*_{kj} and Γ^*_{ij} in (6) are undefined in the interior of the half-space when we choose (ξ_1, ξ_2) to be points on the irregular part of the exterior boundary!

Thus, for $k > 0$; we make the trivial choice $\Phi^*_{kj} = 0$ and $\Gamma^*_{ij} = 0$. With this, however, in discretising (2) and (4), we cannot avoid dealing with the integrals over the intervals $[a, \infty)$ and $(-\infty, -a]$. We supercede these intervals approximately by $[a, 8\ell]$ and $[-8\ell, -a]$, divide the interval $[-8\ell, 8\ell]$ up into many smaller subintervals and approximate the displacements as constants over each of these subintervals. As before, we approximate $\Delta u_i(t)$ using (9). Then, choosing (ξ_1, ξ_2) in (2) to be the midpoint of each of the subintervals and ξ in (4) to be given in turn by (12), we can set up a system of linear algebraic equations to solve approximately for the unknown displacements on the exterior boundary of the half-space and the crack-opening displacements.

As a check for the procedure which we describe above for $\Phi^*_{kj} = 0$ and $\Gamma^*_{ij} = 0$, we use it to recalculate the non-dimensionalised crack energy E/E_0 in Tables 1 and 2. The numerical values obtained by using $M = 10$ and discretising $[-8\ell, 8\ell]$ into 160 subintervals of equal length differ from those in both Tables 1 and 2 by less than 2% (under 1%, in most cases). The difference is practically insignificant.

Table 3: Crack energy E/E_0 vs $k\ell^3 > 0$ for $a/\ell = 1$ and $b/\ell = 0$.

$k\ell^3$	0.100	0.200	0.300	0.400	0.500	1.00	2.00
E/E_0	1.36	1.27	1.23	1.21	1.20	1.17	1.14

For $a/\ell = 1$ and $b/\ell = 0$, we now compute E/E_0 against various values of $k\ell^3 > 0$ for the same transversely-isotropic material, using $M = 10$ and discretising $[-8\ell, 8\ell]$ into 160 subintervals of equal length. The numerical values obtained are given in Table 3. Notice that E/E_0 decreases in magnitude as $k\ell^3$ increases, i.e. the crack is more stable if the "ridge" has a greater height.

Notice that the specific problem with the exterior boundary of the half-space given by (5) and the elastic behaviour of the material governed by (16), as considered above, possesses a certain symmetry about the x_2-axis. According to the symmetry, $\Delta u_2(t) = 0$ (hence $\phi_2^{(m)} = 0$). Also, $u_1^{(1)}(x, f(x)) = -u_1^{(1)}(-x, f(-x))$ and $u_2^{(1)}(x, f(x)) = u_2^{(1)}(-x, f(-x))$ for

$-\infty < x < \infty$. If we wish, we can take advantage of this to reduce the number of unknowns in our approximation and solve a smaller system of linear algebraic equations.

References

[1] Ang, W. T. A hypersingular-boundary integral equation method for a class of multiple interacting crack problems, in PCCE 93 (ed by Kwak, B. M. & Tanaka, M.), pp31-36, *Proceedings of the 1st Pan-Pacific Conf. on Computational Engineering*, Seoul, Korea, 1993, Elsevier Science Ltd.

[2] Ang, W. T. & Noone, G. P. A hypersingular boundary integral equation method for the solution of a multiple interacting crack problem, *Engineering Analysis with Boundary Elements*, 1993, **11**, 33-37.

[3] Chen, W. H. & Chen, T. C. An efficient dual boundary element technique for a two-dimensional fracture problem with multiple cracks, *International Journal for Numerical Methods in Engineering*, 1995, **38**, 1739-1756.

[4] Clements, D. L. *Boundary Value Problems Governed by Second Order Elliptic Systems*, Pitman, 1981.

[5] Clements, D. L. & Haselgrove, M. The boundary integral equation method for the solution of problems involving elastic slabs, *International Journal for Numerical Methods in Engineering*, 1985, **21**, 663-670.

[6] Kaya, A. C. & Erdogan, F. On the solution of integral equations with strongly singular kernels, *Quarterly of Applied Mathematics*, 1987, **45**, 105-122.

[7] Nied, H. F. Periodic array of cracks in a half-space subjected to arbitrary loading, *ASME Journal of Applied Mechanics*, 1987, **54**, 642-646.

[8] Sweeney, J. The stress intensity factors for an edge crack in a semi-infinite orthotropic body, *International Journal of Fracture*, 1988, **37**, 233-241.

[9] Uzel, A. R., Hills, D. A. & Sackfield, A. Stress intensity factors for a cracked half plane, *Journal of Strain Analysis*, 1985, **20**, 209-216.