Inducement of crack propagation

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Abstract

In order to predict fatigue crack growth, an energy conception has been applied. When considering the strain energy in consequence of crack blunting phenomenon, the relation deduced for crack propagation has emerged that it is dependent on both the loading and material parameters. A wide validity of the above expression has been proved. Former calculations on the basis of the plane strain formulation have turned out to be capable of use even in the case of blunt cracks with a small curvature. Close to the front of these cracks, components of both the displacement and stress have been determined by dint of a threedimensional BEM algorithm. Lastly, the stress intensity factor values have been reckoned for variants of a through crack and surface one.

1 Introduction

Works 1. and 2. by D. J. Nicholls demonstrated the fatigue crack propagation to be able to deliver itself with the help of the strain energy released by crack blunting. Making use of this phenomenon in order to predict crack growth appears as an aim of the contribution presented.

There is much arguments in fracture mechanics what make the ideal sharp crack model deficient, among others e.g. in a corrosive environment, it is necessary to generalize the above conception, viz. to introduce a blunting crack tip considering the recurrence of corner, surface and through cracks.

2 Prediction of crack growth in terms with blunting process

It is reasonable to issue from the Griffith’s relation for the strain energy corresponding to an elliptic hole with uniformly distributed load applied at infinity, in the form:
Boundary Elements

\[ U_e = \frac{-B\pi\sigma^2 c^2 \cosh(2\alpha_0)}{2E} \] (1)

where we have denoted by

- \( U_e \) the strain energy released
- \( B \) a constant indicated by the state of stress
- \( \sigma \) the acting stress
- \( c \) the focal interval of the ellipse
- \( \alpha_0 \) an elliptical coordinate

When modelling narrow ellipses, the relation between the variance of strain energy and crack growth increment \( \Delta a \) will be given in the way

\[ \Delta U_e = \frac{B\pi\sigma^2 a}{E} (\Delta a - r) \] (2)

where

- \( a \) - the crack length
- \( r \) - the crack tip radius

There are two types of the strain energy in the foregoing expression: the first term being an energy released by crack extension and the second one connected with crack tip blunting.

To make up the total energy course, it is necessary to involve the surface energy \( \gamma \). The result may also be summarized, as follows:

\[ \Delta U_t = 4\gamma \Delta a - \frac{B\pi\sigma^2 a}{E} (\Delta a + r) \] (3)

while this relation is compatible with Griffith’s expression of the collapse (in accordance with a simple tension). After differentiating Eq. (3) with regard to \( \Delta a \) and putting the result with zero, we get

\[ \frac{d\Delta U_t}{d(\Delta a)} = 0 = 4\gamma - \frac{B\pi\sigma^2 a}{E} \] (4)

that represent Griffith’s condition in the case of instable crack growth. In 1., it was demonstrated that stable fatigue crack growth emerges until the velocity of crack blunting in view of crack extension goes below a critical value down, which is to say
\[
\frac{dr}{d(\Delta a)} = \frac{K_c^2 - K^2}{K^2} \tag{5}
\]

At the same time, the crack tip radius \( r \) may be related to \( \Delta a \) through a polynomial function, e.g. \( r(\Delta a) = c \Delta a^p \), consequently, to express fatigue crack growth, Paris law of the following type results:

\[
\frac{da}{dN} = \left[ p \frac{p}{K_c^2 - \Delta K^2} \right]^{1/(1-p)} \Delta K^{2/(1-p)} \tag{6}
\]

being allowed for variations in the exponent of the law said, in a different way than the bulk of models for fatigue crack growth.

In compliance with A.A. Wells, the stress intensity factor is related to crack opening displacement in terms of the following expression:

\[
\delta = 2r = \frac{\lambda K^2}{E \sigma_y} \tag{7}
\]

For \( \lambda = 1 \) and \( K \ll K_c \), after inserting of the previous relation into Eq. (5), the result reads

\[
\frac{dr}{d(\Delta a)} = \frac{K_c^2}{2r E \sigma_y} \tag{8}
\]

After integrating we get

\[
r = -\frac{K_c^2}{\sqrt{E \sigma_y}} (\Delta a)^{1/2} \tag{9}
\]

Introducing the boundary condition \( \Delta a = 0 \), after some modifications we obtain

\[
K^2 = 2\sqrt{(E \sigma_y \Delta a)} \tag{10}
\]

Eventually suppose \( K_{min} = 0 \) (at that rate \( \Delta K = K \)), i.e. it holds

\[
\Delta a = \frac{da}{dN} = \frac{1}{4E \sigma_y K_c^2} \Delta K^4 \tag{11}
\]
There are next contingencies to determine the stress intensity factor by means of COD conception. In so doing, above all, it is apposite to consider more general implications for both the parameter influenced by partly the state of stress and partly work hardening, and the Young’s modulus differentiated in place stress or strain. E.g., the crack tip opening displacement may be written down in the variant mentioned as follows

$$\delta_1 = \frac{m^{-1/2} K}{E \cdot \delta_y}$$

where $m$ fluctuates from 1 to 2.

It is rigorously stated in 2. that the power law expressing the fatigue crack growth must have the exponent $m$ between 3.5 and 4.5. Values of $m$ for some kinds of alloys are contained in the following chart.

### Table 1. Exponents ($m$) of Paris law being valid for a variety of alloys

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>GRADE STEEL</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alloy steels</td>
<td>15130</td>
<td>1.61</td>
</tr>
<tr>
<td></td>
<td>16326</td>
<td>2.38</td>
</tr>
<tr>
<td></td>
<td>16343</td>
<td>2.34</td>
</tr>
<tr>
<td></td>
<td>19552</td>
<td>2.96</td>
</tr>
<tr>
<td>Heat-resistant steels</td>
<td>17102</td>
<td>5.38</td>
</tr>
<tr>
<td></td>
<td>17153, 17253</td>
<td>4.36</td>
</tr>
<tr>
<td>Alluminium alloys</td>
<td>Al Cu4 Mg1</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>Al Zn6 Mg2 Cu</td>
<td>3.54</td>
</tr>
<tr>
<td>Titanium alloys</td>
<td>analogies: 17246</td>
<td>3.75</td>
</tr>
<tr>
<td></td>
<td>17252</td>
<td>3.96</td>
</tr>
<tr>
<td></td>
<td>17351</td>
<td>3.72-4.01</td>
</tr>
</tbody>
</table>

Hereafter, a comparison of fatigue crack growth data with predictions on the lines of Eq.(11) is shown in Figs 1-2, viz in case of the alluminium alloys. We have elaborated the grades Al Cu4 Mg1 and Al Zn6 Mg2 Cu.
Figure 1: Factual and predicted crack propagation data for Al Cu4 Mg1 alloy

Figure 2: Factual and predicted crack propagation data for Al Zn6 Mg2 Cu alloy
For one of them analogous to the material 2024, the data are consistent with the prediction precisely. In case of the second alloy, analogous with the representant of a series 7000, the greater difference occurs, presumably owing to the circumstance that the interval in given $K_{ic}$ data turned out to be fairly wide. In to the bargain, a good correspondence has been made out between the dates and predictions in 2., namely for stainless steel, titanium and nickel alloys.

3 Modelling effectual parameters of crack problem

Take it that both the upper and the lower surface of the crack, with the exception of a zone at a crack tip, represent planes and the form of the crack tip is a semi-circle of radius $\rho$ for any crack cross-sections. That being so, the distance between the upper and lower crack planes is $2\rho$ (see Fig. 3).

\[
\begin{align*}
\sigma_{zz} & = \frac{K_1}{4\sqrt{2\pi r}} \left[ 5 \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} + \frac{2\rho}{r} \cos \frac{3\theta}{2} \right] \\
\sigma_{nn} & = \frac{K_1}{4\sqrt{2\pi r}} \left[ 3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} - \frac{2\rho}{r} \cos \frac{3\theta}{2} \right] \\
\sigma_{zn} & = -\frac{K_1}{4\sqrt{2\pi r}} \left[ \sin \frac{\theta}{2} - \sin \frac{5\theta}{2} + \frac{2\rho}{r} \sin \frac{3\theta}{2} \right]
\end{align*}
\]

Figure 3: Three-dimensional blunt crack depiction
\[
\sigma_n = \mu \left( \sigma_{nn} + \sigma_{zz} \right) = \frac{2K_1}{\sqrt{(2\pi r)}} \mu \cos \frac{\theta}{2}
\]

\[
\sigma_{zz} = 0
\]

\[
\sigma_{nn} = 0
\]

\[
u = \frac{K_1}{8G} \sqrt{\frac{2r}{\pi}} \left[ (2\kappa + 1) \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} + \frac{2\rho}{r} \sin \frac{\theta}{2} \right]
\]

\[
u = \frac{K_1}{8G} \sqrt{\frac{2r}{\pi}} \left[ (2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} + \frac{2\rho}{r} \cos \frac{\theta}{2} \right]
\]

\[
u = 0.
\] (13)

In so doing, \( \kappa \) differs for plane states - stress and strain case.

Introduce an infinite elastic space with Cartesian coordinates \( \xi \). The j-th component of the traction on a surface with a unit normal vector \( n_k (\xi) \) due to a unit point load acting in the i-th direction at the point \( X \) equals:

\[
T_{ij}(\xi, X) = -\frac{K}{\gamma} \left[ \frac{\delta r}{r^2} \left( \delta_{ij} + \frac{3r_i r_j}{1 - 2\nu} \right) - n_j r_{i,j} + n_i r_{j,i} \right]
\] (14)

where

\[
K = \frac{1 - 2\nu}{8\pi(1 - \nu)}, r = |\xi - X|, \delta_{ij}
\]

\( \delta_{ij} \) - is the Kronecker delta

\( \nu \) - Poisson’s ratio

Differentiations come to pass with respect to \( \xi \).

The relevant j-th displacement component may be specified as:

\[
U_{ij}(\xi, X) = \frac{1}{16\pi Gr(1 - \nu)} \left[ (3 - 4\nu) \delta_{ij} + r_{,i} r_{,j} \right]
\] (15)

The preceding expression holds the Kelvin solution for the unit point load.

Assume that a body takes up part of the space said being bounded by the surface \( A \). After applying both the Kelvin solution and Betti theorem we get a relation in the following form:
\[ u_i(X) = \int_{A} u_j(\xi) T_{ij}^*(\xi, X) dA(\xi) + \int_{A} t_j^*(\xi) U_{ij}^*(\xi, X) dA(\xi) \] (16)

for points X within A. At that, the quantities \( n_j(\xi) \) and \( t_j(\xi) \) are the surface components of both the displacement vector and traction one, respectively.

Suppose that A consists of both surfaces, the external and internal, of a void, while only the surface stresses are given. It is commendable to deduct from the stress field mentioned the part due to the loading applied on the outer surface when the void is missing. This reduced problem is thus load-free on the outside. In this manner, if the outer A grows and becomes unlimited, Eq. (16) keeps its form up when A involves the inner surface merely.

On the lines of \( \Phi \), a plane crack is a special type of void determined by two plane surfaces \( A^\pm \) fixed at \( \xi_3 = 0^\pm \). If the loading is applied on them in a symmetrical way across \( \xi_3 = 0 \), it holds \( t_{ij}^* = -t_{ij}^- \).

For \( T_{ij}^* = -T_{ij}^- \) and \( U_{ij}^* = U_{ij}^- \), Eq. (16) may be modified into the from:

\[ u_i(X) = -\int_{A^*} \Delta u_j(\xi) T_{ij}^*(\xi, X) dA(\xi) \] (17)

where \( \Delta u_j = u_j^* - u_j^- \) and the unit normal vector on \( A^* \) is given by the relation \( n_j^* = -\delta_{j3} \).

After using Hooke’s law, we get the expressions, as follows:

\[ \sigma_{\alpha 3} = G(u_{\alpha 3}^* + u_{3\alpha}^*), \alpha = 1,2 \] (18a)

\[ \sigma_{33} = 2G(\nu u_{\alpha 3}^* + (1-\nu)u_{33}^*)/(1-2\nu) \] (18b)

There is possible to differentiate under the integral sign in Eq. (17) unless \( X \in A^* \). For \( X \rightarrow A^* \), the limits will not exist, and the method of integrations by parts has to be employed.

We have come to the relation

\[ u_{i,\alpha} = -\int_{A^*} \Delta u_j \frac{\partial T_{ij}^*}{\partial X_\alpha} dA = \int_{A^*} \Delta u_j \frac{\partial T_{ij}^*}{\partial \xi_\alpha} = -\int_{A^*} \Delta u_{i,\alpha} T_{ij}^* dA \] (19)

where is denoted
\[
T_{ij}^* = -\frac{K}{r^2} \left[ \frac{X_3}{r} \left( \delta_{ij} + \frac{3r_i r_j}{1-2\nu} \right) + \delta_{j3} r_i - \delta_{i3} r_j \right]
\]

(20)

and \( r_i = (\xi_i - X_i)/r \)

In the quality of the results, the integral equations stand:

\[
\sigma_{a3} = \frac{E(1-2\nu)}{16\pi(1-\nu^2)} \int_{A'} \left[ \sigma_{\beta\alpha} r_{\beta y} - \sigma_{\alpha\nu} r_{\nu y} + \frac{3r_{\alpha} r_{\beta} r_y}{1-2\nu} \right] \frac{\Delta u_{\beta,y} dA}{r^2}
\]

(21a)

\[
\sigma_{33} = \frac{E}{8\pi(1-\nu^2)} \int_{A'} \frac{r_{\alpha} \Delta u_{3,\alpha} dA}{r^2}
\]

(21b)

when indicating the Cauchy’s principal values of the integrals. On the ground of the loading supposed, the shear problem differs from the normal stress one. For all the X the equations afore-said hold, with the exceptions of both the crack contour and alternative singularities of the acting loads.

To determine the stress intensity factors the displacement quantities function, namely at node points that are nearest by the crack tip. In Figs 4-5, SIFs are demonstrated, according to Ref. 3., for blunt crackss both the through and semi-elliptical surface one.

![Figure 4: Stress intensity factor pertinent to through blunt crack](image-url)
4 Conclusion

The entailment of the strain energy released for crack blunting is a likely starting point that may be applied to describe fatigue crack growth while a number of crack opening displacement models appear as utilizable.

The results of the calculation, when employing the algorithm in question, are authentic in case of a small curvature radius of the crack root.

5 References


