A hypersingular numerical Green’s function generation for BEM applied to general fracture mechanics problems - elastodynamics
J. C. F. Telles, L. P. S. Barra and S. Guimaraes
COPPE/UFRJ, Progama de Engenharia Civil, Caixa Postal 68506, CEP21945-970, Rio de Janeiro, Brasil

Abstract
The present paper discusses the application of the Hyper-singular Boundary Integral Equation to obtain the Green’s function solution for elastodynamic Fracture Mechanics problems. The work introduces efficient numerical means of computing the Green’s function components for multiple crack problems, of general geometry. Applications to time harmonic and transient (through inverse numerical transform) problems are presented.

1 Introduction
The implementation of a fundamental solution that includes the crack existence, known as Green’s function, is an important alternative in treating Fracture Mechanics problems by the Boundary Element Method [1].

Originally, the traction (hyper-singular) integral equation was only used [2] to overcome the problem of crack surface degeneration. Since then, the mixed or dual formulation [3, 4], that uses the displacement integral equation over the outer boundary together with the traction integral equation over the crack surfaces to solve all boundary unknowns, has been developed and seems to be well established by now. Once the traction formulation can solve the problem of finding the opening displacement throughout the crack surfaces, the idea of applying this formulation to numerically solve the fundamental Green’s function problem came into a reality [5, 6, 7].

The implementation of the numerical Green’s function (NGF) allows for the solution of fracture mechanics problems in two steps. First the multi-fractured material has to be dealt with to include cracks within the infinite medium, obtaining in numerical fashion a fundamental solution to the second step; then the BEM can be applied to solve the problem over the external body boundary only. Therefore, the existing cracks are previously embedded in the body eliminating the need for integrating over crack surfaces. This provides an accurate BEM solution with smaller systems of equations when compared to other formulations. Another advantage of the proposed strategy is the absence of standard boundary element crack
discretization, specially useful when most of the cracks are small and of simple shape. Inclusions and other material defects could also be added to the fundamental solution, but the main advantage of this approach is its generality allowing for applications to 2 and 3-D problems with equal ease.

In the present work, the NGF approach for Elastodynamic Fracture Mechanics Problems is shown, including time harmonic and transient examples of applications.

2 Numerical Fundamental Green’s Function

As discussed in the first elastostatic applications [5], the fundamental Green’s function can be written in terms of a superposition of a full space fundamental solution plus a complementary part which provides satisfaction of the traction free requirement over the crack surfaces. This Green’s function can be represented by

\[
\begin{align*}
u_{ij}^F(\xi, x) &= u_{ij}^*(\xi, x) + u_{ij}^c(\xi, x) \\
p_{ij}^F(\xi, x) &= p_{ij}^*(\xi, x) + p_{ij}^c(\xi, x)
\end{align*}
\]

where \(u_{ij}^F(\xi, x)\) and \(p_{ij}^F(\xi, x)\) are the fundamental displacement and tractions, in \(j\) direction at the field point \(x\) due to a unit point load with frequency \(\omega\) applied at the source point \(\xi\) in the \(i\) direction. The superscript \(*\) stands for the full space standard fundamental solution (in this case dynamic) and \(c\) indicates the complementary components of the fundamental problem.

Since \(u_{ij}^*(\xi, x)\) and \(p_{ij}^*(\xi, x)\) are known [8], the complementary displacements and tractions, \(u_{ij}^c(\xi, x)\) and \(p_{ij}^c(\xi, x)\), are the unknowns of the fundamental problem. This solution can be written as [5]

\[
\begin{align*}
u_{ij}^c(\xi, x) &= \int_{\Gamma_f} p_{jk}^*(x, \zeta) c_{ik}(\xi, \zeta) \, d\Gamma(\zeta) \\
p_{ij}^c(\xi, x) &= \int_{\Gamma_f} P_{jk}(x, \zeta) c_{ik}(\xi, \zeta) \, d\Gamma(\zeta)
\end{align*}
\]

where \(c_{ik}(\xi, \zeta) = u_{ik}^*(\xi, \zeta S) - u_{ik}^*(\xi, \zeta I) = u_{ik}^F(\xi, \zeta S) - u_{ik}^F(\xi, \zeta I)\) is the crack opening displacements of the Green’s function in which \(S\) and \(I\) stand for ”superior” and ”inferior” surfaces of the crack. The boundary \(\Gamma_f\) represents one of the crack surfaces (’’inferior’’) and, consequently, the sign of the integral depends on this chosen surface of integration.

The integral equation (4) is originated from the hyper-singular or traction formulation. If the crack opening displacements are known, both equations (3) and (4) produce the complementary displacements and tractions at an internal point \(x(x \notin \Gamma_f)\), due to a unit point load of frequency \(\omega\) at \(\xi\). Hence, since the natural boundary condition of the complementary problem is prescribed and given by \(p_{ij}^c(\xi, \zeta) = -p_{ij}^*(\xi, \zeta)\) over \(\Gamma_f\), the limit of equation(4), as \(x \to \Gamma_f\), produces a hyper-singular boundary integral equation for the desired fundamental crack opening displacements. This limiting procedure yields as a final result

\[
\int_{\Gamma_f} P_{jk}^*(\bar{\zeta}, \zeta) c_{ik}(\xi, \zeta) \, d\Gamma(\zeta) = -p_{ij}^*(\xi, \bar{\zeta})
\]
where, the symbol $\int_0^a$ indicates Hadamard’s finite part integral.

In order to simplify the idea and without loss of generality let us consider a particular case of a horizontal crack centred at the origin of the coordinate system. In this case, the components $P_{11}^{\ast}(\zeta, \zeta)$ and $P_{21}^{\ast}(\zeta, \zeta)$ of the dynamic fundamental hyper-singular solution (frequency domain) are null, decoupling the system of equations into longitudinal and transversal crack opening displacement integrals:

\begin{align*}
\int_0^a P_{11}^{\ast}(\zeta, \zeta) c_{11}(\xi, \zeta) \ d\Gamma(\zeta) &= -p_{11}^{\ast}(\xi, \zeta) \\
\int_0^a P_{22}^{\ast}(\zeta, \zeta) c_{12}(\xi, \zeta) \ d\Gamma(\zeta) &= -p_{12}^{\ast}(\xi, \zeta)
\end{align*}

(6)
(7)

where $2a$ is the crack size.

The dynamic fundamental solution is given in terms of Bessel’s functions, approximated by polynomials and logarithmic expressions. The imaginary components of this solution is always regular whereas the real part of the fundamental solution, in the vicinity of $\xi$, i. e., when $r \to 0$, may be written as follows:

\begin{align*}
Re\{P_{11}^{\ast}\} &= \frac{k_0}{r^2} + k_{LRe1} \ln(r) + Re\{P_{11}^{Reg}\} \\
Re\{P_{22}^{\ast}\} &= \frac{k_0}{r^2} + k_{LRe2} \ln(r) + Re\{P_{22}^{Reg}\}
\end{align*}

(8)
(9)

and

\begin{align*}
k_0 &= \frac{G}{2\pi(1-\nu)} \\
k_{LRe1} &= -\frac{G}{4\pi} \left( \frac{c_P^4 + c_S^4}{c_P^2 c_S^2} \right) \\
k_{LRe2} &= -\frac{G}{4\pi} \left( \frac{3c_P^4 - 4c_P^2 c_S^2 + 3c_S^4}{c_P^4 c_S^2} \right)
\end{align*}

(10)
(11)
(12)

where $G$ is the shear modulus, $c_P$ and $c_S$ are compression and shear wave velocities, $\omega$ the loading frequency and $\nu$ Poisson’s ratio. The $P_{11}^{Reg}$ and $P_{22}^{Reg}$ components are regular.

If, instead of frequency domain, the Laplace transform domain is adopted, the singularities of the real part of the fundamental solution are still the same as in equations (8) and (9); i. e., only the coefficients $k_0$, $k_{Re1}$, $k_{Re2}$, $P_{11}^{Reg}$ and $P_{22}^{Reg}$ are different. In addition, the imaginary part of the fundamental solution has a logarithmic singularity which is treated in the same fashion as the ones in equations (8 and 9).

Although the singularity terms differ in each approach, the general treatment is the same for both domains. Equations (6) and (7) can be solved by a standard weighted residual method, using the point collocation technique,
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to produce (manipulating only the equation (6); the procedure for equation (7) is analogous):

\[ \int_a^b P_{11}^*(\tilde{\zeta}_m, \zeta) \ c_{i1}(\xi, \zeta) \ d\Gamma(\zeta) = -p_{i1}^*(\xi, \tilde{\zeta}_m) \quad ; \quad m = 1, 2, \ldots M \quad (13) \]

Allowing for the occurrence of incorrections which can be separately rectified, the standard Gauss quadrature procedure can be employed in equation (13), in the following fashion:

\[ |J| \sum_{n=1}^N \left( P_{11}^*(\tilde{\zeta}_m, \zeta_n) \ c_{i1}(\xi, \zeta_n) W_n \right) - E_{i1} = -p_{i1}^*(\xi, \tilde{\zeta}_m) \quad ; \quad m = 1, 2, \ldots M \quad (14) \]

where \(|J| = a\) is the Jacobian of the transformation to the intrinsic quadrature interval; \(\zeta_n\) the corresponding point at the Gauss station \(n\), \(W_n\) is the associated weighting factor at this station and \(N\) the total number of integration points. The term \(E_{i1}\) is introduced to correct the result of the numerical integral so that the singularity associated differences between finite part and standard numerical integration are counterbalanced.

It is important to note that there is no need to interpolate formally \(c_{i1}(\xi, \zeta_n)\) but simply compute its values at the Gauss’s points, later required for the regular integrals of equations (3) and (4).

In order to define \(E_{i1}\), \(c_{i1}\) can be expanded in Taylor’s series about the point \(\tilde{\zeta}_m\) to extract the singular terms of the integrand. Subtracting the numerical integration of these terms from equation (14) and summing the correct finite part integrals of them, (basically the same procedure of reference [5]), the correction term is obtained to produce the final numerical system of equations (the collocation points are taken to be the same as the Gauss integration ones, \(M = N\)):

\[
a \sum_{n=1}^N \left( P_{11}^*(\tilde{\zeta}_m, \zeta_n) c_{i1}(\xi, \zeta_n) W_n \right) + c_{i1}(\xi, \tilde{\zeta}_m) P_{11}^{Reg}(\tilde{\zeta}_m, \zeta_n)
- c_{i1}(\xi, \tilde{\zeta}_m) \left\{ k_0 \left[ a \sum_{n=1}^N \left( \frac{W_n}{(\zeta_n - \tilde{\zeta}_m)^2} \right) + \frac{2a}{a^2 - \tilde{\zeta}_m^2} \right]
+ k_{LRe1} \left[ a \sum_{n=1}^N \left( W_n \ln(\zeta_n - \tilde{\zeta}_m) \right) \right]
+ 2a - \left( (\tilde{\zeta}_m - a) \ln(\tilde{\zeta}_m - a) + (\tilde{\zeta}_m + a) \ln(\tilde{\zeta}_m + a) \right) \right\}
- \frac{\partial c_{i1}(\xi, \tilde{\zeta}_m)}{\partial x_1(\zeta)} \left\{ k_0 \left[ a \sum_{n=1}^N \left( \frac{W_n}{\zeta_n - \tilde{\zeta}_m} \right) - \ln \left( \frac{a - \tilde{\zeta}_m}{a + \tilde{\zeta}_m} \right) \right] \right\}
+ \frac{ak_0 W_m}{2} \frac{\partial^2 c_{i1}(\xi, \tilde{\zeta}_m)}{\partial \zeta^2} \right. \]

\]
Note that the correct finite part integrals, and principal values, of the singular terms can be calculated either analytically (as presented here) or using any existing appropriate numerical scheme. Crack opening derivatives at \( \tilde{\xi}_m \), present in equation (15), can be obtained by the adoption of a Lagrangean polynomial interpolation function for \( c_{ik}(\xi, \tilde{\xi}_m) \).

Equation (15) can be written in matrix form as

\[
S_1 c_{i1}(\xi) = p_{i1}(\xi)
\]

and, starting from equation (7)

\[
S_2 c_{i2}(\xi) = p_{i2}(\xi)
\]

It should be emphasized that matrices \( S_1 \) and \( S_2 \) are only function of crack geometry. They remain the same for any position of the unit source point load. Hence, the system has to be solved just once, and subjected to back substitutions in a Gauss solution routine for other source points \( \xi \). Therefore, the system matrix is independent of the external boundary shape and discretization. These facts make the implementation quite cost effective and competitive even in the case of simple crack geometries where a close-form Green’s function may be available. Multiple cracks can be considered by letting the integrals over \( \Gamma^I \), in the above expressions, be a sum of integrals over all crack boundaries: \( \int_{\Gamma^I} (...) = \sum_{j=1}^{J} \int_{\Gamma_j^I} (...) \).

The final displacement integral equation is

\[
C_{ij}(\xi) u_j(\xi) = \int_{\Gamma^E} u_{ij}^E(\xi, x) p_j(x) d\Gamma(x) - \int_{\Gamma^E} p_{ij}^E(\xi, x) u_j(x) d\Gamma(x) + \int_{\Gamma_I} c_{ij}(\xi, \zeta) p_j^S(\zeta) d\Gamma(\zeta)
\]

where the last integral represents the crack loading contribution to the external boundary displacements, i.e., crack pressure, if it exists, taking into account that \( p_j^I(\zeta) = -p_j^S(\zeta) \).

The real crack opening displacement of the problem can be obtained by a post-processing procedure, using the standard traction (hyper-singular) integral equation and employing the standard full space fundamental solution.

\[
\int_{\Gamma_I} P_{ij}^*(\xi, \zeta) c_j(\zeta) d\Gamma(\zeta) = \int_{\Gamma^E} P_{ij}^*(\xi, x) u_j(x) d\Gamma(x) - \int_{\Gamma^E} U_{ij}^*(\xi, x) p_j(x) d\Gamma(x) - p_i^S(\xi)
\]

where \( c_j(\zeta) \) is the actual crack opening displacements.
3 Examples

In the following examples, linear continuous elements have been employed throughout the external boundary discretization. The computation of the complementary part of the Green's function has been carried out using 12 Gauss points and the crack length was subdivided in three sub-domains to produce improved representation of the crack opening results in the calculation of the stress intensity factors (SIF) \( K_1 \). This has been computed from crack opening displacements at the Gauss points located at a distance of about 0.025\(a\) from the crack tip. These SIF values are presented in non-dimensional form divided by the static SIF, \( K_0 = \sigma \sqrt{a \pi} \) where \( \sigma \) is the remote applied stress.

3.1 Crack Embedded in an Infinite Medium

The first example consists of a time harmonic plane pressure wave, acting in the normal direction, upon a flat crack in an infinite medium. This problem was previously solved by Mai [9] for plane strains and a wide variety of frequencies. His results permit the comparison presented in Figure (1) where the present NGF solutions are also indicated. In Figure (2) the crack opening amplitudes for some selected frequencies are also compared. The NGF solutions have been computed with only 12 Gauss points for the complete crack.

![K1 x Frequency (P Wave; v=0.25)](image)

Figure 1: Dynamic \( K_1 \) for crack embedded in an infinite medium (\( \gamma = 90^\circ \)).

3.2 Transient Finite Medium Simulation

The transient SIF computation over limited domain problems is, without doubt, a difficult task and many of the most simple problems still do not have a close-form analytical solution. Nevertheless, a solution to the problem indicated in Figure (3) will be discussed here. As can be easily verified,
before external boundary reflections take place (i.e., in the early stage of the analysis), the results to the problem coincide with those of the companion infinite medium problem, with multiple colinear cracks, depicted in Figure (4). In the early stages, however, the latter problem presents results which are the same as the shifted analytical solution to a unique crack within the infinite medium under transient load [10]. This fact permits the comparison presented in Figure (5).

### 3.3 Chen’s Problem

Over the last two decades, the transient problem of a single crack centred in a finite plate (see Figures (6) and (7)) proposed by Chen [11] in 1975 has become a benchmark for this class of analysis. The original solution obtained by Chen was computed using the finite difference technique. Recently, Lin and Ballmann [12] have investigated this problem in detail, also using finite differences, and detected the existence of a peak in the dynamic SIF at around $t = 4 \times 10^{-6}$ s, present in the infinite medium solution. As indicated in Figure (8), this same solution behaviour has been detected in the present implementation.

### 4 Conclusions

The NGF procedure has been extended here to harmonic and transient dynamic SIF computations. The latter is obtained through either Fourier or Laplace’s inverse transform. The accuracy of the solutions indicate that
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Figure 3: Test problem, crack near free surface of a tensioned plate.

Figure 4: Multiple colinear cracks in infinite medium.

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\[ \frac{K_1}{K_0} \]

Figure 5: Time response for \( K_1 \).
Figure 6: Chen's Problem description.

Figure 7: Chen's problem: Discretization.

Rectangular Plate with Centred Crack

K1/K0

Chen (1975)
Lin & Ballmann (1993)
NGF (1996)

Figure 8: Time response for Chen's problem K1.
the good numerical performance of elastostatics is repeated for this more challenging application.

References


