Boundary element treatment of domain integrals for shear deformable plates on elastic foundations

Y. F. Rashed*, M. H. Aliabadi and C. A. Brebbia
Wessex Institute of Technology, Ashurst, Southampton SO40 7AA, UK
*On leave from Structural Engineering Dept., Cairo University, Giza, Egypt

Abstract

In this paper, domain integrals due to a uniform load that appear in the boundary element formulation for shear deformable plates resting on elastic foundations are transformed to equivalent boundary integrals. The Reissner plate bending model is used to model the plate behaviour and both the one-parameter Winkler and the two-parameter Pasternak models are employed to model the behaviour of the foundation. Three numerical examples are presented to demonstrate the accuracy of the present formulation.

1. Introduction

The Boundary Element Method (BEM) has emerged as a powerful numerical method in engineering applications. The main advantage of the BEM is the boundary only discretisation of the body. Usually, when the Euler-type governing differential equation contains non-homogeneous term, domain integrals in the final boundary integral formulation appear. An example to this type of domain integrals is the body force domain integral in elasticity problems [1].

As body force terms are usually prescribed, then the corresponding domain integrals can be transformed to boundary by a single application of the divergence theorem. However, the applicability of the divergence theorem depends on the evaluation of the relevant particular solutions of the associate kernels. In elasticity problems, the so-called Galerkin tensor is used as an equivalent way for indirect evaluation of the required particular solutions [1].

In the application of BEM to Reissner plates resting on two-parameter foundation, Wang et al. [2] derived the fundamental solution and the boundary integral equation. They showed that the fundamental solution has three different cases depending on the plate constants and the foundation parameters.

For an arbitrary plate of thickness $h$ resting on the two-parameter Pasternak foundation model ($k_f$ and $G_f$ represents the two foundation parameters), the following constants can be defined:

$$\alpha = \frac{D k_f + C G_f}{D(C + G_f)} \quad \text{and} \quad \beta = \frac{C k_f}{D(C + G_f)}$$

where $C = D \left(1 - \frac{\nu}{2}\right) \lambda^2$, $D = \frac{E h^3}{12(1 - \nu^2)}$ is the plate flexural rigidity, $E$ is Young's modulus, $\nu$ is Poisson's ratio and $\lambda = \sqrt{10}/h$ is the shear factor.

The characteristic length $\ell$ [3] can be defined as $\ell^4 = 1/\beta$, hence the dimensionless parameter $\kappa = \alpha \ell^2/2$ can be defined.
Boundary Elements

- \( \kappa > 1 \) represent case 1
- \( \kappa = 1 \) represent case 2
- \( \kappa < 1 \) represent case 3

However, the work of Wang et al. [2] is incomplete as they considered only one case of the three possible cases (case 3: \( \kappa < 1 \)). Also, domain integrals were evaluated using a domain discretisation technique. Rashed, Aliabadi and Brebbia [4] derived the other two cases for the fundamental solutions.

In this paper, the domain integrals that appear in the boundary element formulation for Reissner plates resting on two-parameter foundation are transformed to the boundary. The one-parameter Winkler model can be considered as a special case by setting \( G_f \) to zero. The three cases of the fundamental solutions are considered. The transformed kernels exhibit a higher order of singularity; so that a free term appears in the transformed integrals. This free term is affected by the position of the collocation point, that is, internal, external or a boundary point. It is demonstrated that, the hypersingular terms are shown to vanish when integrated along a closed contour. Three numerical examples are presented to demonstrate the accuracy and the behaviour of the transformed boundary integrals.

2. Overview of the relevant domain integrals

In this section, the relevant domain integrals for Reissner plate resting on two-parameter foundation are reviewed. Throughout the paper, the indicial notation is used. Comma denotes derivatives, such as \( (\cdot)_,\alpha = \frac{\partial (\cdot)}{\partial x_\alpha} \) and \( (\cdot),n \) denotes the derivative with respect to the normal \( n \). Greek indices will vary from 1 to 2 whereas, Roman indices from 1 to 3.

Consider an arbitrary plate loaded by a uniform domain load of intensity \( q \) in the \( x_1 \) space. The \( x_1 - x_2 \) plane is assumed to be located at the middle surface \( x_3 = 0 \). The generalised displacement are denoted as \( u_i \), where, \( u_\alpha \) denotes rotations \( (\phi_{x_1} \text{ and } \phi_{x_2}) \) and \( u_3 \) the transverse deflection \( w \) in \( x_3 \) direction. The discretised boundary integral equation has the following form [5]:

\[
[H]{u} = [G]{p} + \{Q\} \tag{2}
\]

where \([H]\) and \([G]\) are the boundary element influence matrices, \( \{u\} \) and \( \{p\} \) are the vectors of the boundary displacements and tractions respectively, and \( \{Q\} \) is the domain load vector which contains the domain integral as follows:

\[
Q_{i3} = \int_{\Omega} U^*_{i3}(x',X)q d\Omega(X) \tag{3}
\]

in which \( U^*_{i3} \) is the two point fundamental solution kernels [4] between the source point \( x' \in \Gamma \) and the field point \( X \in \Omega \).

The moment and shear stress resultants \([S]\) at any internal point can be evaluated by differentiating equation (3) with the respect to the coordinate of the
source point and substitute into the generalized stress-displacement relationships to give [5]:

\[ [S] = ([G])\{p\} - ([H])\{u\} + [Q] \]  

(4)

where \([H]\) and \([G]\) are the derivatives of the boundary element influence matrices \([H]\) and \([G]\) respectively (The symbol \([\cdot]\) represents matrix of sub matrices), and \([Q]\) is defined as:

\[ Q_{\alpha\beta} = \int_{\Omega} U_{\alpha3\beta3}(X', X)q d\Omega(X) \]  

(5)

\[ Q_{33\beta} = \int_{\Omega} U_{33\beta3}(X', X)q d\Omega(X) \]  

(6)

where the new asterisked kernels \((\cdot)^*\) can be found in ref. [5].

3. Transformation of the domain integrals to the boundary

In this section, the domain integral in equation (3) is transformed to the boundary using the divergence theorem. If the particular solution \(V_i^*\) is known such as:

\[ \nabla^2 V_i^* = U_{i3}^* \]  

(7)

then, the following integral identity can be written:

\[ \int_{\Omega} U_{i3}^*(X'', X)q d\Omega(X) = \int_{\Gamma} V_{i,n}^*(X'', x) d\Gamma(x) \]  

(8)

where \(X'' \not\in \Omega\) is exterior collocation point and \(x \in \Gamma\) is a field point. The corresponding particular solutions for the three cases of the fundamental solution as computed by Rashed, Aliabadi and Brebbia [6] as follows:

**Case 1: \((\kappa > 1)\)**

\[ V_{\alpha,n}^* = \frac{-\ell^2}{4\pi D\alpha_3 S} \left[ \frac{1}{r} \left( n_{\alpha} - 2r_{\alpha \tau, n} \right) - \nu_{0,0, r, n} \right] \]

\[ V_{3,n}^* = \frac{-\ell^2 r_{n, n}}{4\pi D\alpha_3 S} \left[ \frac{-2}{(1 - \nu)\lambda^2} \Upsilon_1 + \Upsilon_{-1} \right] \]  

(9)

where

\[ \Upsilon_i = \begin{cases} 
    e^{iK_0(\ell r)} - d^iK_0(d r) & \text{i is even number} \\
    e^{iK_1(\ell r)} - d^iK_1(d r) & \text{i is odd number} 
\end{cases} \]  

(10)

and

\[ d^2 = \frac{\kappa + \sqrt{\kappa^2 - 1}}{\ell^2}, \quad e^2 = \frac{\kappa - \sqrt{\kappa^2 - 1}}{\ell^2}, \quad S = \sqrt{\kappa^2 - 1} \quad \text{and} \quad \alpha_3 = \left( 1 + \frac{G_f}{C} \right) \]  

(11)

in which \(K_0(\cdot)\) and \(K_1(\cdot)\) are modified Bessel functions [7].
226 Boundary Elements

Case 2: \((K = 1)\)

\[
\begin{align*}
V_{\alpha,n}^* &= \frac{-\ell^2}{4\pi D \alpha_3} \left[ A_1 \left( \frac{r}{\ell} \right) (n_\alpha - 2r_\alpha r_n) - (\frac{r}{\ell}) K_1 \left( \frac{r}{\ell} \right) n_\alpha n_n \right] \\
V_{3,n}^* &= \frac{-r_n \ell}{4\pi D (1 - \nu) \lambda^2 \alpha_3} \left[ 4 K_1 \left( \frac{r}{\ell} \right) + \ell^2 \lambda^2 \alpha_4 \left( 2 K_1 \left( \frac{r}{\ell} \right) + (\frac{r}{\ell}) K_0 \left( \frac{r}{\ell} \right) \right) \right]
\end{align*}
\]

where

\[
A_1(\cdot) = K_0(\cdot) + \frac{2}{\ell^2} K_1(\cdot)
\]

and

\[
\alpha_4 = (1 - \nu) - \frac{2}{\ell^2 \lambda^2}
\]

Case 3: \((K < 1)\)

\[
\begin{align*}
V_{\alpha,n}^* &= \frac{-\ell^2}{8D \alpha_3 \sin 2\psi} \left[ \frac{\Psi_{-1}}{r} (n_\alpha - 2r_\alpha r_n) + \Psi_0 n_\alpha n_n \right] \\
V_{3,n}^* &= \frac{-\ell^2 r_n}{8D \alpha_3 \sin 2\psi} \left[ \frac{2}{(1 - \nu) \lambda^2} \Psi_1 + \Psi_{-1} \right]
\end{align*}
\]

where

\[
\Psi_i = \begin{cases} 
2 \text{Re}[\zeta^i H_0^{(1)}(\zeta r)] & \text{i is even number} \\
2 \text{Re}[\zeta^i H_1^{(1)}(\zeta r)] & \text{i is odd number}
\end{cases}
\]

and

\[
\zeta = \frac{e^{i(\frac{\pi}{2} + \psi)}}{\ell}
\]

in which \(\kappa = \cos 2\psi, \sqrt{1 - \kappa^2} = \sin 2\psi, \psi \in [0, \pi/4]\) and \(H_0^{(1)}(\cdot)\) and \(H_1^{(1)}(\cdot)\) are Hankel functions [7].

Consequently the domain terms in equations (5) and (6) can be replaced by boundary integrals for any internal collocation point \(X'\), as:

\[
\int_{\Omega} U_{i\beta}^*(X', X) q d\Omega(X) = q \int_{\Gamma} W_{i\beta}^*(X', X) d\Gamma(X)
\]

where the kernels \(W_{i\beta}^*\) is the derivatives of the kernel \(V_i^*\) with respect to the coordinate of the source point via considering the generalized stress-displacement relationship [5] to give:

\[
\begin{align*}
W_{\alpha\beta}^* &= \frac{D(1 - \nu)}{2} \left[ V_{\alpha,\alpha\beta}^* + V_{\beta,\alpha\alpha}^* + \frac{2\nu}{1 - \nu} V_{\gamma,\gamma\alpha\beta}^* \delta_{\alpha\beta} \right] \\
W_{3\beta}^* &= \frac{D(1 - \nu) \lambda^2}{2} [V_{\beta,\beta\beta}^* + V_{3,\beta\beta}^*]
\end{align*}
\]

The expression for \(W_{i\beta}\) can be found in ref. [6] as follows:
Case 1: \((\kappa > 1)\)

\[
W_{\alpha \beta}^* = \frac{\ell^2}{4\pi \alpha_3 S} \left\{ r_n \gamma_1 [\nu \delta_{\alpha \beta} + (1 - \nu)r_\alpha r_\beta] + \frac{1 - \nu}{r} \left[ 2\gamma_{-1} + \gamma_0 \right] \left( 4r_\alpha r_\beta r_n - \delta_{\alpha \beta} r_n - n_\beta r_\alpha - n_\alpha r_\beta \right) \right\} \\
W_{3\beta}^* = \frac{\ell^2}{4\pi \alpha_3 S} \left\{ \gamma_2 r_\alpha r_n - \frac{\gamma_1}{r} \left[ n_\beta - 2r_\beta r_n \right] \right\} 
\]

(20)

Case 2: \((\kappa = 1)\)

\[
W_{\alpha \beta}^* = \frac{\ell(1 - \nu)}{4\pi \alpha_3} \left\{ \left( \frac{\ell}{r} \right) K_0 \left( \frac{\ell}{r} \right) \left[ \frac{\nu}{1 - \nu} \delta_{\alpha \beta} r_n + r_\alpha r_\beta r_n \right] + \left( \frac{\ell}{r} \right) \left[ 2A_1 \left( \frac{\ell}{r} \right) + \left( \frac{\ell}{r} \right) K_1 \left( \frac{\ell}{r} \right) \right] \left( 4r_\alpha r_\beta r_n - \delta_{\alpha \beta} r_n - n_\beta r_\alpha - n_\alpha r_\beta \right) \right\} \\
W_{3\beta}^* = \frac{1}{4\pi \alpha_3} \left\{ \left( \frac{\ell}{r} \right) K_1 \left( \frac{\ell}{r} \right) r_\alpha r_n - K_0 \left( \frac{\ell}{r} \right) n_\beta \right\}
\]

(21)

Case 3: \((\kappa < 1)\)

\[
W_{\alpha \beta}^* = \frac{-\ell^2}{8\alpha_3 \sin 2\psi} \left\{ r_n \Psi_1 [\nu \delta_{\alpha \beta} + (1 - \nu)r_\alpha r_\beta] - \frac{1 - \nu}{r} \left[ 2\Psi_{-1} \right] \left( 4r_\alpha r_\beta r_n - \delta_{\alpha \beta} r_n - n_\beta r_\alpha - n_\alpha r_\beta \right) \right\} \\
W_{3\beta}^* = \frac{-\ell^2}{8\alpha_3 \sin 2\psi} \left\{ \Psi_2 r_\alpha r_n + \frac{\Psi_1}{r} \left[ n_\beta - 2r_\beta r_n \right] \right\}
\]

(22)

4. Singularity behaviour

Expanding Bessel and Hankel functions for small argument [7], it can be seen that the same singular terms are found in the kernel \(V_{i,n}^*\) for the three cases of the fundamental solution. The singular terms can be written as:

\[
V_{\alpha,n}^* = \frac{-\ell^4}{2\pi D\alpha_3 r^2} (n_\alpha - 2r_\alpha r_n) \\
V_{3,n}^* = \frac{-\ell^4 r_n}{2\pi D\alpha_3 r}
\]

(23)

where the superscript \((\cdot)^*\) denote the singular term of \((\cdot)\).

As it can be seen that \(V_{\alpha,n}^*\) contains a hypersingular term \((O(1/r^2))\) and a strong singular term \((O(r_n/r^2))\); whereas the kernel \(V_{3,n}^*\) contains weakly singular terms of order \((O(r_n/r))\). These high order of singularity results in a discontinuity in the equivalent boundary integrals. These boundary integrals (as
discussed earlier) can be evaluated using Gauss-Legendre formulae when the collocation point is placed outside the boundary. However it is demonstrated by Rashed, Aliabadi and Brebbia [6] that, in the case of boundary collocation the hypersingular terms vanish when integrating them along closed contour. Terms of \(O(r_n/r)\), on the other hand, leads to a jump term. Additional care has to be considered in dealing with internal collocation, as the integrated kernels are not regular over the domain. According to Rashed, Aliabadi and Brebbia [6], equation (8) can be written as:

For boundary collocation point \(x'\)

\[
\int_{\Omega} U_{i3}^*(x', X)q d\Omega(X) = q \int_{\Gamma} \left[ V_{i,n}^*(x', x) - V_{i,n}^*(x', x)(1 - \delta_{i3}) \right] d\Gamma(x) \\
+ \frac{\ell^4 q}{\alpha_3 D} \frac{\varphi_2 - \varphi_1}{2\pi} \delta_{i3} \tag{24}
\]

where \(\varphi_1\) and \(\varphi_2\) are the angle of the tangential with the horizontal before and after the collocation point, respectively.

For internal collocation point \(X'\)

\[
\int_{\Omega} U_{i3}^*(X', X)q d\Omega(X) = q \int_{\Gamma} V_{i,n}^*(X', x) d\Gamma(x) + \frac{\ell^4 q}{\alpha_3 D} \delta_{i3} \tag{25}
\]

It has to be noted that in both boundary and internal collocation procedures, these free terms have no effect on the kernel \(W_{i3}^*\), as they vanish in the differentiation process.

5. Examples

In this section three numerical examples will be considered. The three cases for the formulation will be tested in each example. The numerical integration are performed using the Gauss-Legendre scheme with 10 points for boundary elements and 10 \(\times\) 10 for internal cells.

5.1. Simply supported square plate

In this example, a simply supported square plate of side length \(a\) is considered. Winkler foundation is proposed here \((G_f = 0)\). The plate is loaded uniformly \((q)\). The plate have a thickness of \(0.3a\) and \(\nu = 0.3\). The values of the sub grade parameter \(k_f\) are chosen to give \(\kappa = 1.25, 1.00, 0.75\) in order to represent case 1, 2 and 3 respectively.

Table (1) shows the results of the normalized values of the central deflection and bending moment, as:

\[
\frac{w_c}{w_c} = \frac{w_c D}{qa^4} \times 10^4
\]
and

\[ \overline{M} = \frac{M}{qa^2} \times 10^3 \]

The abbreviations that is used in table (1) stand for the following cases:

- **CB** → results of the domain integrals using \( 4 \times 4 \) constant square cells and boundary collocation.
- **BE** → results of the equivalent boundary integrals using external collocation. Noting that the collocation points are placed outside the boundary by a distance equal to 0.05\( a \).
- **BB** → results of the equivalent boundary integrals using boundary collocation.

In all cases 16 boundary elements were used to model the plate boundary.

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>CB</th>
<th>BE</th>
<th>BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>1.090689</td>
<td>1.090370</td>
<td>1.090388</td>
</tr>
<tr>
<td>1.00</td>
<td>1.735136</td>
<td>1.734837</td>
<td>1.734865</td>
</tr>
<tr>
<td>0.75</td>
<td>3.173222</td>
<td>3.172970</td>
<td>3.172941</td>
</tr>
<tr>
<td>( \overline{w_c} )</td>
<td>0.458920</td>
<td>0.458381</td>
<td>0.458833</td>
</tr>
</tbody>
</table>

Table 1: Simply supported plate results.

As can be seen from table (1) the results of the equivalent boundary integrals are in excellent agreement with that of the domain integrals.

5.2. Clamped circular plate

A circular clamped plate is considered in this example. The plate has a radius \( a \) and a thickness 0.3\( a \). A uniform load \( q \) is applied over the plate domain. Two-parameter pasternak foundation is considered with \( G_F(GFa^2/D) = 20 \) and \( k_F \) is chosen to give \( \kappa = 1.25, 1.00, 0.75 \) in order to represent cases 1, 2 and 3 respectively. It was found that the results of the domain integrals are not affected by the distribution or the number of cells; as the load is applied uniformly. In all of the analysis 16 boundary elements were used. In the (CB) analysis, 24 internal cells were used.

Table (2) show the results for the central point deflection and moment and the radial boundary moment for the circular plate under consideration. In the case of the regular analysis (BE), the collocation points are placed along a fictitious boundary parallel to the plate boundary with a distance equal to 0.05\( a \). It can be seen that the results are in a good agreement.
Table 2: Clamped circular plate results.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa = 1.25$</th>
<th>$\kappa = 1.00$</th>
<th>$\kappa = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{w}_c$</td>
<td>CB 0.526107</td>
<td>BE 0.525870</td>
<td>BB 0.525870</td>
</tr>
<tr>
<td>$\bar{M}_c$</td>
<td>CB 0.006583</td>
<td>BE 0.005828</td>
<td>BB 0.005828</td>
</tr>
<tr>
<td>$\bar{M}_b$</td>
<td>CB -1.494211</td>
<td>BE -1.465514</td>
<td>BB -1.469720</td>
</tr>
</tbody>
</table>

5.3. Comparison between the domain integrals and the equivalent boundary integrals

In this example, the simply supported square plate that was considered in example 1 is studied for this comparison.

The plate is analyzed using both (CB) and (BB) analyses. The value of the integrals $I_\alpha$ and $I_3$ are computed for a different number of Gauss points and at the five collocation nodes shown in Figure 1.

Figures 2 to 4 show a comparison between the normalized values for the domain integrals and the boundary integrals at the five collocation nodes. The normalized integrals are defined as follows:

$$\overline{I_\alpha} = I_\alpha \frac{D}{qa^3} \times 10^4$$

and
\[ \bar{T}_3 = I_3 \frac{D}{qa^4} \times 10^4 \]

As can be seen that the results of \( \bar{T}_3 \) does not affected by the number of Gauss points. Also, the equivalent boundary integrals gives the same accuracy as the using cell discretisation.

The integral \( \bar{T}_3 \), on the other hand, is affected by the number of Gauss points when computed using the cell discretisation. It can be seen that, the value of the integral using the equivalent boundary integrals with 4 Gauss points has the same accuracy as the cell discretisation with 40 Gauss points.

6. Summary and conclusions

In this paper, the domain integrals that appear in the BEM for Reissner plate on two-parameter foundation are transformed to boundary integrals. The technique is based on the application of the divergence Theorem. Three different integrals corresponding to three cases of the fundamental solutions are considered. The necessary particular solutions are evaluated and the relevant kernel derivatives are given. The formulation was validated with three numerical examples. The results show that the equivalent boundary integrals gives accurate results. Two kinds of collocation processes were tested, i.e, boundary and external. In both cases the results show an excellent agreement with that of domain integration.

References


Figure 2: A comparison between the domain integrals and the boundary integrals for different Gauss points-case 1 ($\kappa = 1.25$), (a) for $I_\alpha$ (b) for $I_3$. 
Figure 3: A comparison between the domain integrals and the boundary integrals for different Gauss points-case 2 ($\kappa = 1.00$), (a) for $I_\alpha$ (b) for $I_3$. 
Figure 4: A comparison between the domain integrals and the boundary integrals for different Gauss points-case 3 ($\kappa = 0.75$), (a) for $I_{\alpha}$ (b) for $I_3$. 