Vibrations of plates with variable thickness subjected to inplane forces
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Abstract

A BEM-based method, the Analog Equation Method (AEM), is presented as it is employed to solve dynamic problems for plates of variable thickness subjected to inplane forces. The linear buckling problem is also treated using the dynamic criterium. According to AEM the fourth order hyperbolic partial differential equation with variable coefficients is converted to a quasi-static linear problem for plates with constant stiffness subjected to an appropriate fictitious time dependent load under the same boundary conditions. The fictitious load is established using a technique based on BEM. The method is illustrated by applying it to certain example problems. Linear and exponential plate thickness variation laws are considered.

1 Introduction

The vibration of plates with variable thickness is of great interest in various engineering disciplines, such as civil engineering, aerospace engineering and machine design. Although, there is an extensive literature on vibrations of plates with constant thickness, a rather limited amount of technical literature is available on the solution of problems dealing with vibrations of plates with non-uniform thickness. The reason is due to the fact that the equation describing the response of the plate is a fourth order partial differential equation with variable coefficients. Analytic solutions meet insurmountable difficulties. The existing ones are restricted to plates with simple geometry, rectangular or circular, with simple boundary conditions, and unidirectional thickness variation so that Levy-type solutions can be applied. Numerical methods such as FDM [1], FEM [2] have been employed for the dynamic analysis of plates with variable thickness without inplane forces. The BEM [3] has also been employed. However, with regards to BEM a direct boundary-only method can not be developed since the fundamental solution of the governing equation can not be established. Therefore, the developed BEM solutions are of D/BEM type.
In some recent publications by Katsikadelis and Nerantzaki the AEM has been employed for static [4] and dynamic analysis [5] of plates with variable thickness. Inplane forces have been taken into account only in the static problem with application to buckling [6]. In this paper the AEM is employed to dynamic analysis of plates with variable thickness when inplane forces are present. According to the proposed solution procedure the fourth order hyperbolic partial differential equation with variable coefficients is converted to a quasi-static linear problem for plates with constant thickness subjected to a fictitious time dependant load under the same boundary and initial conditions. The fictitious load is established using a technique based on BEM. The method is illustrated by certain numerical examples.

In this stage the method requires also domain discretization. However, it can be developed to a boundary-only method using the procedure presented in Ref.[7].

2 Governing equations

Consider a thin elastic plate of variable thickness, \( h = h(x, y) \), occupying the two-dimensional multiply connected region \( \Omega \) of the \( x, y \) plane. The equilibrium of a plate element subjected to a distributed transverse load \( g(x, t) : \{x, y\} \in \Omega, t \geq 0 \) and inplane forces \( N_x = N_x(x), N_y = N_y(x) \) and \( N_{xy}(x) \) yields the following differential equation of motion in terms of the deflection \( w(x, t) \).

\[
D \nabla^4 w + 2 \frac{\partial D}{\partial x} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial D}{\partial y} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \]

\[-(1 - \nu) \left( \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) \]

\[-(N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}) + c \frac{\partial w}{\partial t} + \rho \frac{\partial^2 w}{\partial t^2} = g(x, t) \tag{1} \]

where \( D = E h^3 / 12(1 - \nu^2) \) is the variable flexural rigidity of the plate, \( c = c(x) \) the distribution of the damping coefficient and \( \rho \) the mass density. Moreover, the deflection \( w \) must satisfy the following boundary conditions on the boundary \( \Gamma \) and initial conditions inside \( \Omega \)

\[
\alpha_1 w + \alpha_2 \dot{V}(w) = \alpha_3 \quad \beta_1 \frac{\partial w}{\partial n} + \beta_2 \dot{M}(w) = \beta_3 \quad \text{on} \ \Gamma \tag{2a,b} \]

\[
w(x, 0) = \bar{w}(x), \quad \frac{\partial w(x, 0)}{\partial t} = \dot{\bar{w}}(x) \quad \text{in} \ \Omega \tag{3a,b} \]
where \( \alpha_i = \alpha_i(x,t) \), \( \beta_i = \beta_i(x,t) \), \( x \in \Gamma \), are functions specified on \( \Gamma \); \( M(w) \) and \( V(w) \) are the bending moment and the reactive force on the boundary; \( \bar{\omega}(x) \) and \( \bar{\omega}(x) \) are the initial deflection and the initial velocity of the points of the middle surface of the plate, respectively.

Taking into account that the flexural rigidity \( D \) is a function of the variables \( x \) and \( y \) and using intrinsic co-ordinates \( n \) and \( s \), the operators \( M, V \) appearing in eqns (2a,b) may be written as

\[
M = -D[\nabla^2 + (\nu - 1)(\frac{\partial^2}{\partial s^2} + \kappa \frac{\partial}{\partial n})] 
\]

\[
V = -D[\frac{\partial}{\partial n} \nabla^2 - (\nu - 1) \nabla^2 \frac{\partial}{\partial s} \left( \frac{\partial}{\partial n} \right) + 2(\nu - 1) \frac{\partial}{\partial s} \left( \frac{\partial^2}{\partial n^2} - \kappa \frac{\partial}{\partial n} \right) - \nabla^2 \frac{\partial}{\partial n} \right]
\]

in which \( \kappa = \kappa(s) \) is the curvature of the boundary; \( \partial / \partial s \) and \( \partial / \partial n \) denote differentiation with respect to the arc length \( s \) of the boundary, and the outward normal \( n \) to it, respectively.

The stress resultants at a point inside \( \Omega \) are given as

\[
M_x = -D \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad M_y = -D \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad M_{xy} = D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} 
\]

\[
Q_x = -D \nabla^2 w - \frac{\partial}{\partial x} \nabla^2 \left( \frac{\partial w}{\partial x} + \nu \frac{\partial w}{\partial y} \right) - (1 - \nu) \frac{\partial}{\partial y} \frac{\partial^2 w}{\partial x \partial y} 
\]

\[
Q_y = -D \nabla^2 w - \frac{\partial}{\partial y} \nabla^2 \left( \frac{\partial w}{\partial y} + \nu \frac{\partial w}{\partial x} \right) - (1 - \nu) \frac{\partial}{\partial x} \frac{\partial^2 w}{\partial y \partial x} 
\]

Since the linear problem is considered, the inplane forces \( N_x, N_y, N_{xy} \) are \textit{a priori} known. They are given as

\[
N_x = h\sigma_x = Ch \left( \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right), \quad N_y = h\sigma_y = Ch \left( \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) 
\]

\[
N_{xy} = N_{yx} = C \frac{1 - \nu}{2} h \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) 
\]

in which \( C = E / (1 - \nu^2) \) and \( u = u(x,y) \), \( v = v(x,y) \) are the inplane displacement components which are established by solving independently the
plane stress problem for plates with variable thickness, which, in absence of body forces, is described by the following Navier-type differential equations

\[
\mu h \nabla^2 u + (\lambda + \mu) h \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial h}{\partial x} \left[ \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2 \mu \frac{\partial u}{\partial x} \right] + \frac{\partial h}{\partial y} \left[ \lambda \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2 \mu \frac{\partial v}{\partial y} \right] = 0 \quad (7a)
\]

\[
\mu h \nabla^2 v + (\lambda + \mu) h \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial h}{\partial x} \left[ \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2 \mu \frac{\partial u}{\partial x} \right] + \frac{\partial h}{\partial y} \left[ \lambda \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2 \mu \frac{\partial v}{\partial y} \right] = 0 \quad \text{in } \Omega \quad (7b)
\]

under the boundary conditions

\[
u = \bar{v}, \quad \nu = \bar{v} \quad \text{on } \Gamma_1, \quad \quad \quad \text{u} = \bar{u}, \quad f_y = \bar{f}_y, \quad \text{on } \Gamma_2 \quad (8a,b)
\]

\[
u = \bar{v}, \quad f_x = \bar{f}_x, \quad \text{on } \Gamma_3, \quad f_x = \bar{f}_x, \quad f_y = \bar{f}_y, \quad \text{on } \Gamma_4 \quad (8c,d)
\]

where \( \bigcup_{i=1}^{4} \Gamma_i = \Gamma \) and \( \lambda \) and \( \mu \) are the Lamé constants.

The boundary forces \( f_x \) and \( f_y \) are given in terms of the displacements as

\[
f_x = Ch[(u_x + \nu v_y) \cos \alpha + \frac{1 - \nu}{2} (u_y + v_x) \sin \alpha] \quad (9a)
\]

\[
f_y = Ch[(\nu u_x + v_y) \sin \alpha + \frac{1 - \nu}{2} (u_y + v_x) \cos \alpha] \quad (9b)
\]

The boundary value problem (7a,b), (8a)-(8d) is solved using the FEM or the AEM [8].

3 The analog equation method (AEM)

The initial-boundary value problem described by eqns (1),(2) and (3) is solved using the AEM. In the problem at hand this method is applied as follows.

Let \( w \) be the sought solution of eqn (1). This function is four times continuously differentiable with respect to the spatial co-ordinates \( x, y \) in \( \Omega \). If the biharmonic operator is applied to this function we have

\[
\nabla^4 w = q(x,t) \quad (10)
\]

Eqn (10) indicates that the solution of the original initial-boundary value problem can be obtained as the solution of a linear quasi-static bending problem for a plate with unit stiffness and subjected to the equivalent (fictitious) time-dependent load \( q \) under the given boundary and initial conditions.

According to the AEM, the unknown load \( q \) can be established using the BEM. The direct BEM for plates could be applied if the boundary terms in
Rayleigh-Green identity were modified so that to include the boundary reaction defined by eqn (4b). However, this procedure is avoided because it would involve complicated singular and hypersingular kernels which would be difficult to manipulate and evaluate numerically. Therefore, an indirect BEM developed by Katsikadelis & Armenakas [9] has been employed in the development of AEM because of the simplicity of the kernels appearing in the boundary integrals.

According to this method, for any function \( w \) satisfying the non-homogeneous biharmonic eqn (10) the following integral representations are obtained

\[
e w(x, t) = \int_\Omega \Lambda_4 q d\Omega - \int_{\Gamma} (\Lambda_1 \Omega + \Lambda_2 X + \Lambda_3 \Phi + \Lambda_4 \Psi) ds
\]

\[
\varepsilon \nabla^2 w(x, t) = \int_{\Omega} \Lambda_2 q d\Omega - \int_{\Gamma} (\Lambda_1 \Phi + \Lambda_2 \Psi) ds
\]

where \( \varepsilon = 2\pi, \pi, 0 \) depending on whether the point \( x \) is inside the domain \( \Omega \), on the boundary \( \Gamma \) or outside \( \Omega \), respectively. Note that the boundary has been assumed to be smooth at the point \( x \). The kernels \( \Lambda_i = \Lambda_i(r) \), with \( |\xi - x| \), and \( x \in \Omega, \xi \in \Gamma \) corresponding to the fundamental solution of eqn (10) are given as

\[
\Lambda_1(r) = -\frac{r}{r}, \quad \Lambda_2(r) = \ell nr + 1
\]

\[
\Lambda_3(r) = -(2\ell nr + r)n_r/4, \quad \Lambda_4(r) = r^2 \ell nr/4
\]

On the basis of eqns (4a,b) the boundary conditions (2a,b) are written as

\[
\alpha_1 w + \alpha_2 \left\{-D[\Psi - (\nu - 1) \frac{\partial}{\partial \kappa} \left( \frac{\partial}{\partial \kappa} - \kappa \frac{\partial}{\partial \kappa} \right)] + 2(\nu - 1) \frac{\partial^2}{\partial \kappa^2} \left( \frac{\partial}{\partial \kappa} - \kappa \frac{\partial}{\partial \kappa} \right) \right\} = \alpha_3
\]

\[
\beta_1 \frac{\partial w}{\partial n} - \beta_2 D[\Phi + (\nu - 1) \frac{\partial^2}{\partial \kappa^2} + \kappa \chi)] = \beta_3
\]

In eqns (11),(12),(14) and (15) the following notation has been used

\[\Omega = w(x), \ X = w_n(x), \ \Phi = \nabla^2 w(x), \ \Psi = \nabla^2 w_n(x) \quad x \in \Gamma\]

The integral representations (11) and (12) for \( x \in \Gamma \), together with the boundary conditions (14) and (15) constitute a set of four boundary equations with respect to the boundary quantities \( \Omega, X, \Phi, \Psi \). Two of these equations are boundary integral equations and the remaining boundary differential equations.
They are solved numerically. The boundary differential equations are solved using the finite difference method, whereas the integral equations are solved using the BEM. Constant boundary elements are employed. The domain integrals are evaluated using constant triangular or rectangular cells.

The above discretization yields the following set of linear equations.

\[
\begin{bmatrix}
[A_{11}] & [A_{12}] & [A_{13}] & [A_{14}] & \{\Omega\} \\
[A_{21}] & [A_{22}] & [A_{23}] & [0] & \{X\} \\
[A_{31}] & [A_{32}] & [A_{33}] & [A_{34}] & \{\Phi\} \\
[0] & [0] & [A_{43}] & [A_{44}] & \{\Psi\}
\end{bmatrix}
\begin{bmatrix}
\{B_1\} \\
\{B_2\} \\
\{0\} \\
\{0\}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
[C_3] \\
[C_4]
\end{bmatrix} = \{q\}
\]

(17)

where \([A_{ij}]\) \((i, j = 1, 2, 3, 4)\) are \(N \times N\) known coefficient matrices originating from the integration of the kernels on the boundary elements, \({B_i}\) \((i = 1, 2)\) known constant vectors and \([C_i]\) \((i = 3, 4)\) \(N \times M\) known coefficient matrices originating from the integration of the kernels over the domain cells. \(\{\Omega\}, \{X\}, \{\Phi\}, \{\Psi\}\) are vectors including the \(N\) nodal values of the unknown boundary quantities, while \(\{q\}\) is a vector including the \(M\) values of the unknown fictitious loading at the nodal points inside \(\Omega\). From eqns (17) the boundary quantities \(\Omega, X, \Phi, \Psi\) are expressed in terms of the fictitious load vector \(\{q\}\). Subsequently, their substitution into the discretized counterpart of eqn (11) yields when it is applied to the nodal points inside \(\Omega\)

\[
\{w\} = [G]\{q\}
\]

(18)

where \(\{w\}\) is a vector including the values of the deflection \(w\) at the \(M\) domain nodal points and \([G]\) is an \(M \times M\) known coefficient matrix.

Subsequent differentiation of eqn (11), when \(s = 2\pi\), gives

\[
2\pi w_{xx}(x, t) = \int_R (\Lambda_4)_{xx} q d\Omega - \int_\Gamma [(\Lambda_1)_{xx} \Omega + (\Lambda_2)_{xx} X + (\Lambda_3)_{xx} \Phi + (\Lambda_4)_{xx} \Psi] ds
\]

(19a)

\[
2\pi w_{yy}(x, t) = \int_R (\Lambda_4)_{yy} q d\Omega - \int_\Gamma [(\Lambda_1)_{yy} \Omega + (\Lambda_2)_{yy} X + (\Lambda_3)_{yy} \Phi + (\Lambda_4)_{yy} \Psi] ds
\]

(19b)

\[
2\pi w_{xy}(x, t) = \int_R (\Lambda_4)_{xy} q d\Omega - \int_\Gamma [(\Lambda_1)_{xy} \Omega + (\Lambda_2)_{xy} X + (\Lambda_3)_{xy} \Phi + (\Lambda_4)_{xy} \Psi] ds
\]

(19c)

\[
2\pi \nabla^2 w_x(x, t) = \int_R (\Lambda_2)_{x} q d\Omega - \int_\Gamma [(\Lambda_1)_{x} \Phi + (\Lambda_2)_{x} \Psi] ds
\]

(19d)

\[
2\pi \nabla^2 w_y(x, t) = \int_R (\Lambda_2)_{y} q d\Omega - \int_\Gamma [(\Lambda_1)_{y} \Phi + (\Lambda_2)_{y} \Psi] ds
\]

(19e)

The derivatives of the kernels are given in Ref. [5].
Eliminating the boundary quantities from the discretized counterparts of eqns (19a)-(19e) by means of eqns (17) and collocating at the $M$ nodal points inside $\Omega$ yields

$$\begin{align*}
\{w_{xx}\} &= \{G_{xx}\}\{q\}, \\
\{w_{yy}\} &= \{G_{yy}\}\{q\}, \\
\{w_{xy}\} &= \{G_{xy}\}\{q\}
\end{align*}$$

(20a,b,c)

$$\begin{align*}
\{\nabla^2 w_x\} &= \{G_{Lx}\}\{q\}, \\
\{\nabla^2 w_y\} &= \{G_{Ly}\}\{q\}
\end{align*}$$

(20d,e)

where $[G_{xx}], [G_{yy}], [G_{xy}], [G_{Lx}], [G_{Ly}]$ are known $M \times M$ coefficient matrices. Note that eqns (16) and (20a)-(20e) are valid for homogeneous boundary conditions ($\alpha_3 = \beta_3 = 0$). For non-homogeneous boundary conditions an additive vector will appear in the right hand side of these equations.

The final step of the AEM is to apply eqn (1) at the $M$ nodal points inside the domain $\Omega$. This yields

$$\begin{align*}
[D]\{\nabla^4 w\} + 2[D_x]\{\nabla^2 w_x\} + 2[D_y]\{\nabla^2 w_y\} + [\nabla^2 D]\{\nabla^2 w\} \\
- (1-v)\{[D_{xx}]\{w_{xx}\} - 2[D_{xy}]\{w_{xy}\} + [D_{yy}]\{w_{yy}\}\} \\
- [N_x]\{w_{xx}\} + 2[N_{xy}]\{w_{xy}\} + [N_y]\{w_{yy}\} + \{c]\{\ddot{w}\} + \{\rho h\}\{\ddot{w}\} = \{g\}
\end{align*}$$

(21)

Substituting eqns (10),(18) and (20a)-(20e) in eqn (21), we obtain

$$\begin{align*}
[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [S]\{q\} = \{g\}
\end{align*}$$

(22)

where $[M],[C]$ and $[S]$ are known square matrices given by

$$\begin{align*}
[M] &= [\rho h][G], \\
[C] &= [c][G], \\
[S] &= [K] - [B]
\end{align*}$$

(23a,b,c)

$$\begin{align*}
[K] &= \{[D] + 2[D_x][G_{Lx}] + 2[D_y][G_{Ly}] + [\nabla^2 D][G_{xx} + [G_{yy}]\} \\
- (1-v)\{[D_{xx}]\{G_{yy}\} - 2[D_{xy}]\{G_{xy}\} + [D_{yy}]\{G_{xx}\}\} \\
[B] &= \{[N_x][G_{xx}] + 2[N_{xy}][G_{xy}] + [N_y][G_{yy}]\}
\end{align*}$$

(24a,b)

Note that the square matrices including $\rho h$, $c$ as well as $D$ and its derivatives are diagonal matrices.

Eqn (22) is the semidiscretized equation of motion of the plate with variable thickness with $[M],[C]$ and $[S]$ representing the generalised mass, damping and stiffness matrices, respectively. Its solution yields $\{q\}$. The initial conditions for eqn (22) are obtained using eqn (16) as

$$\begin{align*}
\{q_o\} &= [G]^{-1}\{\ddot{w}\}, \\
\{\dot{q}_o\} &= [G]^{-1}\{\dot{w}\}
\end{align*}$$

(25a,b)
For free undamped vibrations, \( c = 0 \), \( g = 0 \), and by taking \( q(x, t) = Q(x)e^{-i\omega t} \) eqn (22) yields the following equation

\[
(-\omega^2[m] + [s])[Q] = [0]
\]

from which the eigenfrequencies and mode shapes are established numerically by solving the generalised eigenvalue problem. In evaluating the domain integrals in the discretized counterparts of eqns (11), (12) and (19a)-(19e) we come across to kernels, which behave as \( \ln r \), \( 1/r \) and \( 1/r^2 \) for small values of the argument \( r \). Thus, we have to evaluate singular and hypersingular domain integrals on the internal cells. This can be effectively done by converting the domain singular integrals into regular ones on the boundary of the cell using Green’s reciprocal identity [10].

4 Numerical Examples

On the base of the procedure described in the previous section a computer program has been developed to establish the eigenfrequencies and modes shapes of plates with variable thickness subjected to inplane forces. In all example treated the numerical results have been obtained using 60 constant boundary elements and 100 rectangular domain cells. The eigenfrequencies have been computed using the subroutine DGVLRG of MSIMSL.

A square simply supported plate with side \( \alpha \), Poisson’s ratio \( \nu = 0.30 \) under uniform compression in the \( x \) direction, \( N_x \) = constant, \( N_y = N_{xy} = 0 \), has been studied.

The thickness variation law has been taken:

(i) Constant.

(ii) Linear \( h = h_0(1 + \beta x / \alpha) \), \( \beta = h_\alpha / h_0 - 1 \), \( 0 \leq x \leq \alpha \)

(iii) Exponential \( h = h_0\exp(\frac{x}{\alpha} \ln \frac{h_\alpha}{h_0}) \), \( 0 \leq x \leq \alpha \)

The variation of \( \Omega = \alpha(\omega^2 \rho / D_0)^{1/4} \) versus \( \bar{N}_x = N_x\alpha^2 / (\pi^2 D_o) \), \( D_o = Eh_0^3 / 12(1 - \nu^2) \), for \( h_\alpha / h_o = 1.25 \) are shown in Figure 1. The values of \( (\bar{N}_x)_\sigma \), where the curves \( \Omega = f(\bar{N}_x) \) cross the \( \bar{N}_x \) axis, are the buckling loads. These values when compared with those from analytical or other numerical solutions [6] are found in very good agreement. From Figure 1, one can conclude that the difference between linear and exponential thickness variation is negligible for \( h_\alpha / h_o = 1.25 \).
Figure 1.
202 Boundary Elements

References


