A generalized viscoplastic constitutive model with a mixed strain hardening for implicit BEM

Y. Liu*, J. Bridgwater** and H. Antes***

*Dept. of Mechanical Engineering, Zhejiang University of Technology, Hangzhou, 310014, P.R. China
**Dept. of Chemical Engineering, The University of Cambridge, Cambridge CB2 3RA, UK
***Institute of Applied Mechanics, TU Braunschweig, D-38106, Germany

Abstract

This paper presents a generalized constitutive model with a mixed strain hardening for implicit elastic/visco-plastic boundary element method, which includes both the isotropic and kinematic hardenings. As compared with the usual implicit scheme, in this implicit scheme, the viscoplastic strain rate contains not only the current stress increment but also the viscoplastic strain increment. Numerical example, discussion and comparison with existing research results are presented to illustrate the performance of the implicit algorithm.

1. Introduction

The elastic/viscoplastic material model which is suitable for many materials has widely used in FEM and BEM for a long time [1-6]. In these work, applications of the elastic/visco-plasticity theory in engineering commonly use an implicit or explicit scheme. The implicit scheme, which permits larger time steps to be taken, is proved to be more stable than explicit scheme. This algorithm is fully described by [7] for ideal plasticity in FEM, and later by [8] and [9] in BEM. But there are few references to discuss the incorporation of a general strain-hardening model in the implicit elastic/viscoplastic algorithm. Ref.[10] presented an implicit algorithm for the inclusion of strain hardening/softening of implicit elastic/viscoplasticity in FEM, but, it is only suitable for the general isotropic strain hardening case. In this paper, a generalized constitutive model with mixed strain hardening for implicit algorithm of viscoplastic BEM is presented. Both the isotropic and kinematic strain hardening effects are included in the scheme as the special cases and the viscoplastic strain rate contains not only the current
stress increment but also the viscoplastic strain increment. Finally, numerical example, discussion and comparison with existing research results are presented to illustrate the performance of the implicit algorithm.

2. The generalized elasto-viscoplastic constitutive model for implicit algorithm

It is known that the general viscoplastic strain increment scheme in the time interval \( \Delta t_n = t_{n+1} - t_n \) is defined to be

\[
\Delta \varepsilon_{vp}^n = \Delta t_n \left[ (1 - \alpha^*) \dot{\varepsilon}_{vp}^n + \alpha^* \dot{\varepsilon}_{vp}^{n+1} \right] \quad (1)
\]

in which the viscoplastic strain rate is (for associate case)

\[
\dot{\varepsilon}_{vp} = \gamma < \Phi(F) > \frac{\partial F}{\partial \sigma} \quad (2)
\]

where \( F \) is the yield function, \( \Phi(F) \) is a flow function, and \( \gamma(t) \) is a time variable fluidity coefficient. The appropriate choice of \( \alpha^* \) in the range \( 0 \leq \alpha^* \leq 1 \) defines various explicit and implicit time integration schemes. \( \alpha^* = 0 \) represents the forward Euler scheme, i.e., the explicit scheme; \( 0 < \alpha^* \leq 1 \) represents the implicit scheme.

Owen and Hinton [7] derived a direct solution method for the implicit algorithm by expressing \( \dot{\varepsilon}_{vp}^{n+1} \) as a limited Taylor's series

\[
\dot{\varepsilon}_{vp}^{n+1} = \dot{\varepsilon}_{vp}^n + H_n \Delta \sigma_n \quad (3)
\]

in which \( \Delta \sigma_n \) is the stress increment, and

\[
H_n = \left[ \frac{\partial \dot{\varepsilon}_{vp}^T}{\partial \sigma} \right]_n = \gamma \left\{ \left\langle \frac{d\Phi}{dF} \right\rangle a a^T + \left\langle \Phi \right\rangle \frac{\partial a^T}{\partial \sigma} \right\} \quad (4)
\]

is the viscoplastic strain derivative matrix, \( a = \frac{\partial F}{\partial \sigma} \) is the flow vector. Substituting eqn (3) into eqn(1) leads to the usual implicit algorithm.

For a static and or quasi-static problem, the yield function \( F \) takes the following forms with strain hardening

\[
F = F(\sigma, J_2^*, J_3^*, \kappa(\varepsilon_{vp}), t) = F(\sigma, \varepsilon_{vp}) \quad (5)
\]
which becomes function not simply of the stress but also of the generalized visco-plastic strain. Now, let’s reconsider the implicit visco-plastic strain increment scheme (1). The visco-plastic strain $\dot{\varepsilon}_{vp}^{n+1}$ can be expressed as

$$
\dot{\varepsilon}_{vp}^{n+1} = \gamma \Phi_{n+1}(F) a_{n+1}
$$

The first-order Taylor series expansion of $F(\sigma, \varepsilon_{vp})$ becomes

$$
F(\sigma_{n+1}, \varepsilon_{vp}^{n+1}) \equiv F(\sigma_n, \varepsilon_{vp}^n) + \frac{\partial F^T}{\partial \sigma_n} \Delta \sigma_n + \frac{\partial F^T}{\partial \varepsilon_{vp}^n} \Delta \varepsilon_{vp}^n
$$

considering $F$ as a function of the stress and the visco-plastic strain as independent variables. Supposing $\frac{\partial F^T}{\partial \varepsilon_{vp}^n} = c_n^T$, the increment of the yield surface can be written as

$$
\Delta F_n = F_{n+1} - F_n = a_n^T \Delta \sigma_n + c_n^T \Delta \varepsilon_{vp}^n
$$

By using the Taylor series expansions of the flow function $\Phi$

$$
\Phi_{n+1} \approx \Phi_n + \Phi(F)(F_{n+1} - F_n)
$$

and inserting eqs (8-9) in the flow rule eqn (6) and disregarding the second order terms, one can obtain the approximation:

$$
\dot{\varepsilon}_{vp}^{n+1} \approx \dot{\varepsilon}_{vp}^n + H_n \Delta \sigma_n + G_n \Delta \varepsilon_{vp}^n
$$

where

$$
G_n = \gamma \Phi_n \frac{\partial a_n^T}{\partial \varepsilon_{vp}^n} + \gamma \Phi_n a_n c_n^T
$$

One can see from eqn (10), it contains both the current stress increment and viscoplastic strain increment which is different from the previous solution (3).

Substituting eqn (10) into the time discretization scheme (1) and solving it for $\Delta \varepsilon_{vp}$ yields

$$
\Delta \varepsilon_{vp}^n = \Delta t_n Q_n \dot{\varepsilon}_{vp}^n + \alpha^* \Delta t_n Q_n H_n \Delta \sigma_n
$$

where $Q_n = \left(I - \alpha^* \Delta t_n G_n \right)$. If neglecting the strain effect, i.e., $C_n^T = 0$, and $\frac{\partial a_n^T}{\partial \varepsilon_{vp}^n} = 0$ lead to the matrix $G_n = 0$, the completely pure elastic/visco-plastic implicit scheme is obtained. Thus, eqn (12) is a generalized improved implicit
scheme for an elastic visco/plastic strain hardening model. As an application, a mixed strain hardening model [11] is considered. There, assuming the yield function \( F(\sigma, \varepsilon_{vp}) \) takes the form

\[
F(\sigma, \varepsilon_{vp}) = f(\sigma_y - \alpha_y) - h(\varepsilon_{vp}^{(i)}) = 0
\]  

(13)

where \( \alpha_y \) denotes the translation of the centre of the yield surface, \( h \) is a function which governs the isotropic expansion or contraction of the yield surface, and \( \varepsilon_{vp}^{(i)} \) is an effective visco-plastic strain quantity governing the process of isotropic hardening. Their expressions are

\[
\alpha_y = \beta (1 - M) \varepsilon_{vp}
\]

\[
h(\varepsilon_{vp}^{(i)}) = h(M \varepsilon_{vp})
\]  

(14)

where \( \beta = \frac{2}{3} H' \), \( H' = \frac{\partial h}{\partial \varepsilon_{vp}} \), and \( M \) is a material parameter in the range

\[-1 \leq M \leq 1
\]  

(15)

defining the share of the isotropic hardening in the total amount of hardening, for \( M=1 \), the isotropic hardening holds, whereas with \( M=0 \), the kinematic hardening is obtained. Since the material parameter \( M \) can be given negative values, strain softening can also be considered.

Substituting eqs (13) and (14) into eqn (10), one obtains the following visco-plastic strain rate:

\[
\dot{\varepsilon}_{vp}^{n+1} = \dot{\varepsilon}_{vp}^n + H_n^M \Delta \sigma_n + G_n^M \Delta \varepsilon_{vp}^n
\]  

(16)

where, \( H_n^M = H_n(\sigma_y - \alpha_y, h) \), \( G_n^M = \gamma \Phi_n \frac{\partial a_n^T}{\partial \varepsilon_{vp}} + \gamma \Phi'_n a_n \left( \frac{\partial f}{\partial \varepsilon_{vp}} - M H' m \right) \)

\[
m^T = \frac{\partial \varepsilon_{vp}^n}{\partial \varepsilon_{vp}^n} = \frac{1}{3} \left( \frac{2}{3} \varepsilon_{vp}, \frac{2}{3} \varepsilon_{vp}, \frac{2}{3} \varepsilon_{vp}, \frac{1}{3} \gamma_{xy}, \frac{1}{3} \gamma_{xz}, \frac{1}{3} \gamma_{yz} \right)
\]

Eqn (16) is the mixed hardening (softening) strain rate expression. Inserting it into eqn (1), one can obtain the mixed strain hardening implicit scheme, which is the same form as eqn (12)

When \( M=1.0 \), i.e., the isotropic case holds, one has
\[ F = f(\sigma) - h(\varepsilon_{vp}) = 0, \quad \frac{\partial a_n^T}{\partial \varepsilon_{vp}} = 0, \quad \frac{\partial f}{\partial \varepsilon_{vp}} = 0 \] (17)

\[ H_n^M = H_n \left[ \sigma, h(\varepsilon_{vp}) \right], \quad G_n^M = \gamma \Phi_n \frac{\partial a_n^T}{\partial \varepsilon_{vp}} - \gamma \Phi_n' a_n m^T \]

in which, if neglecting the strain effect, i.e. \( \frac{\partial a_n^T}{\partial \varepsilon_{vp}} = 0 \), one can obtain the same form as ref [10].

When \( M = 0.0 \), i.e., the kinematic strain hardening case is obtained:

\[ F = f(\sigma, \varepsilon_{vp}) - h = 0, \quad h = \sigma_s = \text{constant}, \quad \frac{\partial h}{\partial \varepsilon_{vp}} = 0 \]

\[ H_n^M = H_n \left( \sigma_{ij} - \alpha_{ij} \right), \quad G_n^M = \gamma \Phi_n \frac{\partial a_n^T}{\partial \varepsilon_{vp}} + \gamma \Phi_n' a_n \left( \frac{\partial f}{\partial \varepsilon_{vp}} \right)^T \] (18)

3. The elasto-viscoplastic boundary element approach

For the solution of a general elasto/viscoplastic problem, a boundary integral equation can be obtained through weighted residual procedures or in terms of the reciprocal theorem (neglecting the body force)

\[ c_{ij} (\xi) \dot{u}_j (\xi) = \int_{\Gamma} u^*_j (\xi, x) p_j (x) d\Gamma (x) - \int_{\Gamma} p^*_j (\xi, x) \dot{u}_j (x) d\Gamma (x) + \int_{\Omega} \sigma_{jkl}^* (\xi, x) \dot{\varepsilon}_{vpkl} (x) d\Omega (x) \] (19)

where \( u^*_j, p^*_j \) and \( \sigma^*_{jkl} \) are displacement, traction and stress fundamental solutions, and \( \dot{u}_j, \dot{p}_j, \dot{\varepsilon}_{vpkl} \) are the displacement, traction and viscoplastic strain rate components of problem to be solved; \( \Omega \) represents the domain of the body, \( \Gamma \) is its boundary and \( C_{ij} \) is the coefficient found in elastic analysis.

The internal stress rates can be written as:

\[ \dot{\sigma}_{ij} (\xi) = \int_{\Gamma} u_{ij}^* (\xi, x) \dot{p}_k (x) d\Gamma (x) - \int_{\Gamma} p_{ij}^* (\xi, x) \dot{u}_k (x) d\Gamma (x) + \int_{\Omega} \sigma_{ijkl}^* (\xi, x) \dot{\varepsilon}_{vpkl} (x) d\Omega (x) + f_{ij} (\dot{\varepsilon}_{vpkl}) \] (20)

where the stress fundamental solution \( \sigma_{ijkl}^* \) and the free term \( f_{ij} \), which introduce the viscoplastic influence, are given in [12], \( \dot{\sigma}_{ij} \) represents the interior stresses.
Application of the discretization procedure to all boundary and interior nodes for eqs (19) and (20) generates the following increment form of matrix relationship

\[ \Delta y = K \Delta \varepsilon_{vp} + \Delta m_1 \]
\[ \Delta \sigma = B \Delta \varepsilon_{vp} + \Delta m_2 \]

(21)

where, \( \Delta y \) represents the unknown quantities, \( \Delta m_1 \) and \( \Delta m_2 \) are prescribed terms of tractions and boundary displacements, \( K \) and \( B \) are the stiffness matrices. Eqn (21) can be solved with the generalized viscoplastic implicit scheme (12) by standard "initial strain" method (see [6], [8], [12]).

Now, let's check eqn (12), which can be expressed as

\[ \Delta e_{vp} = \Delta e_{vp} + \bar{C} \Delta \sigma \]

(22)

where \( \Delta e_{vp} = Q \dot{\varepsilon}_{vp} \Delta t, \bar{C} = \alpha^* \Delta t Q \ H \)

Eqn (22) when substituted in the incremental form of the second equation of eqn (21), leads finally to

\[ C^{vp} \Delta \sigma = \Delta \bar{n} \]

(23)

in which \( C^{vp} = I - K\bar{C}, \Delta \bar{n} = B \Delta e_{vp} + \Delta m_2 \)

Eqn (23) has the same form as obtained by [8]. But, here, \( C^{vp} \) is a viscoplastic matrix with a generalized strain hardening model, which is different from the results of [8]. It is clear that for \( \alpha^* = 0 \), eqn (23) reduces to the simple Euler procedure, i.e., the explicit mixed strain hardening scheme.

4. Numerical example

A simply supported deep beam under uniform load is studied by applying the improved implicit viscoplastic boundary element method. The BEM discretization employed is shown in Fig.1 (for symmetric case, only a half is considered). The material is assumed to obey the Tresca, Mohr-Coulomb and Zienkiewicz-Pande (hyperbola) criteria with the following parameters

\[ E = 30 \times 10^6 \text{ psi}, \sigma_s (or c) = 36 \times 10^3 \text{ psi}, \varphi = 40^\circ \]
\[ \nu = 0.3, \ H' = 0.0 \text{ or } \frac{\sigma_s}{2} (\text{for hardening}), \gamma = 1.0 \text{ sec}^{-1}, \alpha^* = 1.0 \]
\[ \tau = 0.05, \Delta t_0 = 0.001, k = 1.5, l = 16 \text{ in} \]

for plane stress, the thickness of the beam is taken as 1.0 in.
The problem has been analyzed by Anand et al [13] by using a mesh of 272 linear displacement triangular finite elements, and Telles, Brebbia et al [4],[12] with 36 boundary elements and 68 internal triangular elements. Here, we use 18 boundary elements and 12 eight-noded quadrilateral interior cells. A comparison of results without strain hardening is shown in Fig.2 and Tab 1 under Tresca’s criterion. As can be seen, the boundary element solution of this paper is very close to the limit load obtained by the collapse beam theory (see Tab.1), whereas the finite element results slightly exceed this load level.
Table 1 Comparison the collapse loads of beam theory and BEM

<table>
<thead>
<tr>
<th></th>
<th>Beam theory [39]</th>
<th>BEM</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{P_{\text{collapse}}}{\sigma_s} )</td>
<td>0.125</td>
<td>0.124164</td>
<td>0.67%</td>
</tr>
</tbody>
</table>

The comparison with three different yield criteria is shown in Fig. 3. Fig. 3 gives the load-mid deflection curves with strain hardening, in which the highest load is taken as \( \frac{P}{\sigma_s} = 0.13 \). It can be seen that the value of Zienkiewicz-Pande yield criterion reaches the highest value, which is due to the coincidence of its surface with the outer apices of the Mohr-Coulomb hexagon in principle stress space. In this figure, a comparison with Tresca criterion (no-hardening) is also included, which shows a little lower value than the results with strain hardening effect.

![Fig. 3 Load-midspan displacement curves with three yield criteria](image-url)

Table 2 Comparison stabilities with strain hardening (Tresca criterion, \( H' = 18 \times 10^6 \) psi)

<table>
<thead>
<tr>
<th>( \gamma ) ( \Delta t )</th>
<th>0.001</th>
<th>0.01</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implicit scheme ( \alpha^* = 1.0, M=0.5 )</td>
<td>stability</td>
<td>stability</td>
<td>stability</td>
</tr>
<tr>
<td>explicit scheme ( \alpha^* = 0.0, M=0.5 )</td>
<td>stability</td>
<td>instability</td>
<td>instability</td>
</tr>
</tbody>
</table>
Tab.2 gives the comparison of the stability with strain hardening for both the implicit and explicit scheme. The iteration results of explicit scheme are stable at $\gamma \Delta t = 0.001$, whereas, when $\gamma \Delta t$ takes larger values, e.g., $\gamma \Delta t = 0.01$ and 0.05, the results are unstable. However, the results of the implicit algorithm are all stable at the three cases. This shows again that the implicit viscoplastic scheme can take larger time step lengths than explicit schemes. The iteration number and the stability at different load increment are also calculated in this example. The results show that the iteration number with strain hardening is less than that without strain hardening, especially when $\gamma \Delta t$ takes larger value.

5. Conclusions

This paper proposes a generalized implicit viscoplastic boundary element algorithm for the strain-hardening problems, in which the mixed strain hardening model, including both the isotropic and kinematic strain hardenings, is presented. A generalized matrix $G$ corresponding to the viscoplastic strain of the implicit scheme is developed, which are particular suitable for strain hardening analyses. As compared with previous implicit schemes, in the present improved implicit scheme, the viscoplastic strain rate contains not only the current stress increment but also the viscoplastic strain increment. Numerical example has shown that the problems involving mixed strain hardening behaviour in the algorithm of this paper has not only a marginal effect on stress and displacement results during loading phase, but also some effects on iteration number and stability, as compared with an implicit viscoplasticity algorithm in which the strain-hardening is ignored.

Acknowledgements: The support by the Alexander von Humboldt Foundation is gratefully acknowledged.

References


