A simple coupling of 2D BEM and FEM bar model applied to mass matrix elastodynamic analysis

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Abstract

In this work a finite element bar is coupled to a body modelled by two-dimensional Boundary Element Method. The coupling is made for both static and mass dynamic problems. Examples are presented to demonstrate accuracy and efficiency of the proposed formulation.

Introduction

In recent years much progress has been made in coupling the finite elements with the boundary elements. A review of different techniques can be found in Beskos and Stamos[1] and Beer[2].


In this paper a simple technique for connecting a finite element bar to a two-dimensional medium modelled by the boundary element is presented. The BEM is formulated by a transient mass approach using the static fundamental solutions (see for example [10,11,12]). This approach is presented here again with a view of analysing finite regions reinforced by means of linear bars modelled by FEM.
Three examples are presented to demonstrate the applicability of the formulation.

**Dynamic BEM mass approach formulation**

The starting point of the BEM dynamic mass approach for analysis of elastic and homogeneous body is the generalised Maxwell-Betti reciprocal theorem applied to two independent displacement states \( \{u, \sigma\} \) and \( \{u', \sigma'\} \) [12] written as:

\[
\int_\Omega [u_i \sigma_{ij} - u'_{ij}] d\Omega = \int_\Gamma [u_i p_i - u_{ij} p_{ij}] d\Gamma
\]  

(1)

where the Einstein notation is applied and \( u_i \) represents the components of displacements in the \( i \) direction, \( \sigma_{ij} \) is the stress tensor and \( p_i \) are the surface traction given by \( p^i = \sigma^i n^j \), where \( n \) is the outward normal vector related to the surface \( \Gamma \) of the body \( \Omega \).

The dynamic equilibrium equation can be written as:

\[
\sigma^i_{k} + b_i = \rho \dddot{u}_i + a v \dot{u}_i
\]  

(2)

where \( b_i \) is the body force, \( \rho \) is the mass density, \( \dddot{u}_i \) is the acceleration of the related point, \( a v \) is the viscous damping parameter and \( \dot{u}_i \) is the velocity.

To obtain the direct mass BEM formulation, a particular elastostatic state, known as the Kelvin’s fundamental solution is utilised. This fundamental solution can be derived from equation (2) when it represents the static response of an infinite domain \( \Omega^* \) subjected to a concentrated unit impulse, that is

\[
\sigma_{kij} + \delta(s, q) \delta_{ki} = 0
\]  

(3)

where \( k \) is the direction of the applied concentrated force, \( \delta(s, q) \) is Dirac Delta function, \( s \) is the load application point, \( q \) is a field point and \( \delta_{ki} \) is the Kronecker Delta. From the solution of the above differential equation the fundamental solutions for displacement \( u^*_{i} \) and tractions \( p^*_{ki} \) are obtained [13].

Substituting the fundamental solutions into equations (1) and using equation(2) , we obtain the following integral representation:

\[
\int_{\Omega} \int_{\Gamma} \int_{\Omega} C_{ki} u_i(S) = \int_{\Gamma} u^*_{ki}(S, Q)p_i(Q)d\Gamma(Q) - \int_{\Gamma} u_i(Q)p^*_{ki}(S, Q)d\Gamma(Q)
\]

\[
+ \int_{\Omega} b_i(q)u^*_{ki}(S, q)d\Omega(q) - \rho \int_{\Omega} \dddot{u}_i(q)u^*_{ki}(S, q)d\Omega(q) - av \int_{\Omega} \dot{u}_i(q)u^*_{ki}(S, q)d\Omega(q)
\]  

(4)

where \( C_{ki} \) is a constant that depends only on the geometry at the source point \( s \) and can be computed as shown in reference [14].

In order to achieve the system of time differential equations, suitable to numerical treatment the boundary of the body is discretized into quadratic isoparametric boundary elements. The last three domain integrals are performed using isoparametric eight nodes quadratic cells.

Beyond this standard discretization, internal linear load lines are introduced to make possible the coupling of immerse finite element bars. These load lines are constructed as a couple of lines, symmetrically located in relation to their composing internal nodes as shown in figure 1.
The distance \( h \) is defined by the height of the finite element bar introduced inside the body. The internal lines are treated as boundary elements computing only the first integral of eqn (1). This procedure avoids displacement approximations and the necessity of closure on the top of the first and the last load line as well as at angular connections. As the composing internal nodes (see fig. 1) can be very close to the lines, the resulting quasi-singular integrals are computed by an accurate algorithm, based on a progressive element subdivision technique [15].

Taking into account what have been described above, the resulting system of equation can be written as

\[
H \ddot{U} + C \dot{U} + M \ddot{U} = B + G P + F \quad (5)
\]

Where \( M \) and \( C \) are respectively the mass and the viscous dumping matrices, \( B \) is the volume forces and \( F \) is concentrated forces inside the body.

Equation (5) is a standard system of time differential equations and can be numerically integrated in time by, for example, the Newmark \( \beta \) method [16].

\[
\begin{align*}
M + \frac{1}{2} (\Delta t)C + \beta (\Delta t)^2 H \right] \dot{U}_{s+1} &= (\Delta t)^2 \left[ \beta G P_{s+1} + (1 - 2\beta) G P_s + \beta G P_{s-1} \right] + \\
&+ (\Delta t)^2 \left[ \beta (B + F)_{s+1} + (1 - 2\beta) (B + F)_s + \beta (B + F)_{s-1} \right] \\
&+ \left[ 2M - (\Delta t)^2 (1 - 2\beta) H \right] U_s - \left[ M - \frac{1}{2} (\Delta t) C + \beta (\Delta t)^2 H \right] U_{s-1}
\end{align*}
\]

where \( s + 1 \) is the instant in which the unknowns are calculated, the other values are known as they are past values. After solving an instant the resulting values become past for the next step and this procedure continues until the end of the analysis.

Equation (6) can be rewritten in a compact form as follows:

\[
\overline{H} \ddot{U} = \overline{G} P + A \quad (7)
\]

in which the vector \( A \) contains independent values and \( U \) and \( P \) are the actual variables of the problem.

To impose boundary conditions in equation (7) the columns of \( \overline{H} \) and \( \overline{G} \) are interchanged as it is usual for the BEM analysis.

### Finite Element Formulation

The finite element method for elastic bodies, under static or dynamic loads, is briefly discussed in this section. The formulation presented is based on the Virtual Work Principle (see for example [17,18,19]).
Considering a generic body under static loads, the Virtual Work Principle can be written as:

\[
\int_{\Omega} \varepsilon_{ij} \sigma_{ij} \, d\Omega = \int_{\Omega} \ddot{u}_k b_k \, d\Omega + \int_{\Gamma} \ddot{u}_k p_k \, d\Gamma + \dddot{u}_k^i F^i
\]  

(8)

where \( \varepsilon \) is the strain tensor, the upper bars represent virtual displacements, the subscripts represent directions and the superscripts represent defined positions.

To obtain the Finite Element algebraic system the body is divided into internal elements. The discretised form of (8) can be written in a matrix form as

\[
KU = B + GP + F
\]

(9)

where \( K \) is the stiffness matrix, \( U \) and \( F \) represent the displacement and node force vectors, \( B \) represents the equivalent body force nodal values and the product \( GP \) usually called equivalent nodal distributed force values is here written in its extended form to make possible the consideration of surface forces as unknowns. In this sense \( G \) matrix is called the consistent lumping matrix.

To obtain the dynamic equilibrium equation D’Alambert principle is applied in equation (8) to give

\[
\int_{\Omega} \varepsilon_{ij} \sigma_{ij} \, d\Omega + \int_{\Omega} \ddot{u}_k \rho \dddot{u}_k \, d\Omega + \int_{\Gamma} \dddot{u}_k a_n \dddot{u}_k \, d\Gamma = \int_{\Omega} \ddot{u}_k b_k \, d\Omega + \int_{\Gamma} \ddot{u}_k p_k \, d\Gamma + \dddot{u}_k^i F^i
\]

(10)

After performing all integration over all elements the usual system of time differential equations for dynamic problems is represented as follows,

\[
KU + C\dot{U} + M\ddot{U} = B + GP + F
\]

(11)

where \( M \) and \( C \) are respectively the mass and damping matrices; the values \( \dot{U} \) and \( \ddot{U} \) are respectively the velocity and acceleration vectors.

The body to be analysed by the FEM formulation, in this work, is a simple bar following the Bernoulli Hypothesis. Cubic approximation for transversal displacement and linear approximation for the longitudinal one are adopted. The distributed surface load is assumed to be linear in the longitudinal direction as have been made for the BEM internal load element. The nodal variables are two translations and one rotation at each node.

As the stiffness, mass and dumping matrices are usually found in the literature, here only the lumping matrix, \( G \), is given, that is

\[
G = \begin{bmatrix}
\frac{L}{3} & 0 & 0 & \frac{L}{6} & 0 & 0 \\
0 & \frac{7L}{2} & -1 & 0 & \frac{3L}{2} & -1 \\
0 & \frac{20}{L} & \frac{20}{L} & 0 & \frac{12}{L} & -1 \\
0 & \frac{20}{L} & \frac{12}{L} & 0 & \frac{30}{L} & 12 \\
0 & \frac{L}{6} & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{3L}{20} & \frac{20}{L} & 0 & \frac{1}{2} & 2 \\
0 & \frac{L}{30} & \frac{L}{12} & 0 & \frac{L}{20} & \frac{1}{L}
\end{bmatrix}
\]

(12)

where \( L \) is the length of the bar.
The time step integration algorithm used to solve equation (11) is the same used for the BEM analysis, changing the matrix $H$ by the matrix $K$. In this sense the reduced form given in equation (7) is valid.

**Coupling**

In order to couple BEM with FEM, the subregion technique is used. Considering two subregions defined by $\Omega_i$ and $\Omega_j$, coupled together through an interface $\Gamma_{ij}$, one can apply equation (7) for both bodies resulting:

$$H^i U^i = G^i P^i + A^i \quad (13)$$
$$H^j U^j = G^j P^j + A^j \quad (14)$$

The equilibrium and geometrical compatibility conditions at the interface $\Gamma_u$ can be written as:

$$U^{ij} = U^{ji} \quad (15)$$
$$P^{ij} = -P^{ji} \quad (16)$$

where the superscripts denote the first and the second subregions at the contact.

The values $U^{ij}$ and $P^{ij}$ are respectively displacement and surface forces at the contact area. The exterior values for both subregions, i.e., values that do not belong to the contact surface, are called $U^{xe}$ and $P^{xe}$. When these values are introduced in equations (13) and (14), we have

$$\begin{bmatrix} H^{xe} & H^{ij} \\ 0 & H^{ji} \end{bmatrix} \begin{bmatrix} U^{xe} \\ U^{ij} \end{bmatrix} = \begin{bmatrix} G^{xe} & G^{ij} \\ 0 & G^{ji} \end{bmatrix} \begin{bmatrix} P^{xe} \\ P^{ij} \end{bmatrix} + \begin{bmatrix} A^i \\ A^j \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} H^{xe} & H^{ij} \\ 0 & H^{ji} \end{bmatrix} \begin{bmatrix} U^{xe} \\ U^{ij} \end{bmatrix} = \begin{bmatrix} G^{xe} & G^{ij} \\ 0 & G^{ji} \end{bmatrix} \begin{bmatrix} P^{xe} \\ P^{ij} \end{bmatrix} + \begin{bmatrix} A^i \\ A^j \end{bmatrix} \quad (18)$$

Taking into account equations (15) and (16), expressions (18) and (19) can be coupled together to give

$$\begin{bmatrix} H^{xe} & H^{ij} & -G^{ij} & 0 \\ 0 & H^{ji} & G^{ji} & H^{xe} \end{bmatrix} \begin{bmatrix} U^{xe} \\ U^{ij} \\ P^{xe} \\ P^{ji} \end{bmatrix} = \begin{bmatrix} G^{xe} & G^{ij} & 0 & 0 \\ 0 & 0 & G^{xe} & G^{ji} \end{bmatrix} \begin{bmatrix} P^{xe} \\ P^{js} \\ F^{ije} \\ F^{jii} \end{bmatrix} + \begin{bmatrix} A^i \\ A^j \end{bmatrix} \quad (19)$$

where $F^{ij}$ denotes prescribed surface force at the contact surface. The above expression can be generalised for more subregions [20]. The matrices that appear in expression (19) contain full and null blocks, and to solve the resulting system of equation some numerical procedures are available in the specialised literature.

**Examples**

In the first example a finite element beam is coupled with a BEM rectangular domain as shown in figure 2. The material properties are the same for both subregions and given as

$$E = 210 \times 10^7 \text{kg} / \text{(dm s}^2) \; ; \; \nu = 0.2 \; .$$

The momentum of inertia adopted for the horizontal FEM bars is $I = 2.25 \text{dm}^4$, for the vertical (or contact) ones it is varied
from $I = 0.25 \text{ dm}^4$ to $I = 900 \text{ dm}^4$. The adopted discretization is shown in figure 2. The contact tractions, shear and normal, are shown respectively in figures 3a and 3b.
As it is expected the contact tractions tend to the simplified Bernoulli hypothesis as the contact becomes more rigid.

The second example is the simulation of a clamped beam subjected to a suddenly applied transversal load, figure 4. The material properties are: $E = 21 \text{GPa}$; $\nu = 0.2$, $\rho = 2500 \text{kg/m}^3$ and $\alpha v = 100 \text{kg/(m}\cdot\text{s})$. This analysis is carried out to demonstrate the accuracy of the 2D dynamic BEM mass approach when compared with a simple bar finite element procedure. Beyond this, it also demonstrated that the BEM solutions are not sensitive to the mass discretization (see figure 5). The adopted time step is $\Delta t = 0.0005 \text{s}$. The results for vertical displacements at the free end of the bar are shown in figure 6.

![Figure 4: Geometry and load behaviour.](image)

![Figure 5: Used discretizations](image)

![Figure 6- Vertical displacement of the free end.](image)
In the third example, the same beam is analysed, but now it is introduced an immerse finite element bar in order to reinforce the continuous medium. This reinforcement is accomplished by freedom the locked displacement at the tensioned part of the bar at the fixed end, see figure 7. The material properties assumed for the immerse bar are: $E = 210\text{GPa}$ and $\rho = 7000\text{kg/m}^3$. The adopted reinforcing bar transversal area are variable and its values are shown in figure 8. It is considered no influence of the reinforcing bar momentum of inertia. In figure 8, the results of the reinforced beam are compared with the result of the first example, named fixed continuum. In figure 9 the more deflected shape of the reinforced and the continuum beams are compared.

![Geometry and discretization of the problem.](image)

Figure 7: Geometry and discretization of the problem.

![Displacement at the free end of the reinforced bar](image)

Figure 8: Displacement at the free end of the reinforced bar.
As it can be seen from figure 8, the reinforcement of the bar brings the displacement to the level of the continuous one, but the frequency of the movement is not affected as expected. The shapes exhibited at figure 9 confirm that the frequencies should not be the same. The reinforced one presents a large slope rotation at its support end accomplished by a more rigid shape, what conduces to more action of the mass influence than the continuum one, leading to a lower vibration frequency.

Conclusions

The coupling of a finite element bar with a 2D dimensional continuum body modelled by the mass boundary element method for both static and dynamic problems have been successfully implemented. Examples presented demonstrate that this formulation is very general and useful for practical purposes. The principal improvements to be introduced in this formulation are: more general load line to make more precise the definition of curved reinforcements and the physical non-linear analysis. The principal utility of this formulation will be the coupling of it, working in the non-linear branch, with the transient formulation for the region where the non-linearities do not occurs.

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References

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