Boundary element method for transient problems of uncoupled thermoelastodynamics

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Abstract

The non-stationary boundary value problems of uncoupled thermoelastodynamics are solved. Two variants of the boundary integral equations are constructed, one of them in Laplace time-transformation space. For displacements definition the indirect potential method was used. The temperature field is determined on the base of direct method. The computer realisation of BEM is demonstrated on examples of external boundary value problem for plane with arched holes by the action of impulsive heat flow. BIE in original space with use of retarding potential also are considered and there particulars are discussed.

1 Introduction

The researches of boundary value problems in thermoelastic mediums with use of boundary element method (BEM) was first used for stationary (static) processes independent of time (t). The quasistatic problems are more complex class with temperature (Θ) dependent on t and the static equations of elasticity where the body forces depend on the gradient of Θ(x,t) are used in the definition of the stress-strain state. Such problems were studied by V.Sladek and J.Sladek\(^1\) with use of BEM for plane case and G.E.Dargush, P.K.Banerdjee\(^2,3\) in plane and three-dimensional cases. The third class contains time-dependent problems, when the acceleration of displacement \(u(x,t)\) is taken into account in motion equations. The BEM for such problems is developed recently . Apparently the first BE analysis of linear coupled thermoelasticity plane problems with Laplace time-transformation were published by I.J.Suh, N.Tasaka\(^4\).

This paper is devoted to this class of dynamic problems. Here the model of uncoupled thermoelastodynamics is considered. For investigation of stress-strain state of thermoelastic media by the action of heat flow at its boundary it’s appropriately to use the model of uncoupled thermoelastics, because deforma-
tions in a body are slow in this case and stipulated by them temperature variations are more less then ones under boundary heat flow action. It gives a possibility to define the temperature separately, without regard the body deformations.

The boundary integral equations (BIE) are constructed in Laplace time-transformation space. The determination of inverse transformations are performed by use of numerical methods. For displacements definition in uncoupled model the indirect potential method was used which had been developed by us for decision of plane problems of elastodynamics \(^5\). The direct method was used for definition of temperature field. It succeeded to obtain the relations containing temperature instead of its derivatives, what is more convenient by calculations. The computer realisation of this method is demonstrated on example of external boundary value problem for plane with arched hole under the action of non-stationary bounding heat flow.

In the case of BIE in original space only the theoretical aspects of BEM are elaborated. In particular by using of distribution theory the dynamic analogy of Somigliana formulæ and BIE were constructed for thermoelastic displacements and temperature.

2 The statement of a problem

Thermoelastic medium occupies a domain \(D\) with smooth finite boundary \(S\), \(D \subset \mathbb{R}^2\), \(n\) is unit vector of external normal to \(S\). The components of displacement \(u_i(x,t)\), stress tensor \(\sigma_{ij}(x,t)\) and temperature \(\Theta(x,t)\) satisfy to the equations:

\[
\sigma_{ij,j} + X_i = \rho u_i, , X_i = -\gamma \Theta_{,t}, \tag{1}
\]

\[
\Theta_{,ij} - \frac{1}{k} \Theta_{,t} + Q = 0, \quad i, j = 1, 2, \tag{2}
\]

\[
\sigma_{ij} = (\lambda u_k, k - \gamma \Theta) \delta_{ij} + \mu (u_i, j + u_j, i).
\]

The initial conditions are null:

\[u(x,0) = 0, \quad x \in D + S; \quad u_t(x,0) = 0, \quad x \in D; \quad \Theta(x,0) = 0, \quad x \in D + S.\]

The loads and thermal flow on the boundary \(S\) are known:

\[
\sigma_{ij}(x,t) n_i(x) = f_i(x,t), \quad \frac{\partial \Theta(x,t)}{\partial n(x)} = q(x,t), \quad x \in S, \quad t > 0.
\]

It’s require to determine the thermal stress-strain state with use of BEM.

3 BIE of uncoupled thermoelastodynamics in Laplace transformation space

Here BIE are constructed in a space of Laplace transformation of time \(t\). The transformant of temperature \(\tilde{\Theta}(x,p)\) can be expressed over its bounding value and heat flow \(^6\).
Here $U(x,y,p) = -K_0(\sqrt{p/\kappa})$, $K_0(z)$ is Macdonald's function, $H_D(z)$ is the characteristic function of $D$:

$$H_D(x) = \begin{cases} 1, & x \in D \\ 1/2, & x \in S \\ 0, & x \notin D + S \end{cases}$$

For $x \in S$ Eq. (3) gives BIE to define $\theta(x,p)$ which is solved here by numerical methods.

The displacements are presented as a sum of solutions of homogeneous and inhomogeneous equations of elastodynamics:

$$\bar{u}(x,p) = \bar{u}^1(x,p) + \bar{u}^2(x,p)$$

$$\bar{u}^1_i(x,p) = \int S \bar{U}_{ij}(x,y,p)\varphi_j(y,p) dS(y)$$

$$\bar{u}^2_i(x,p) = -\gamma \int_D \bar{U}_{ij}(x,p) \ast (\bar{\theta}(x,p) H_D(x)) \cdot j =$$

$$= -\gamma \int_D \bar{U}_{ij} \cdot j(x-y,p) \bar{\theta}(y,p) dV(y)$$

$$= \frac{\gamma p}{2\pi c_0^3} \int_D \bar{\theta}(y,p) K_1 \left( \frac{pr}{c_1} \right) \frac{\partial r}{\partial y_i} dV(y)$$

Here $\bar{U}_{ij}(x,p)$ is the transformant of Green's tensor $U_{ij}(x,t)$ of elastodynamics equations with singular body force $X_i = \delta(x)\delta(t)\delta^j_i$, where $\delta^j_i$ is Kronecker symbol:

$$\bar{U}_{ik} = \frac{p}{2\pi \mu} \left( \psi(r,p) \delta_{ik} - \chi(r,p) r_i r_k \right)$$

$$\psi(r,p) = K_0 \left( \frac{pr}{c_2} \right) + \frac{c_2}{pr} \left( K_1 \left( \frac{pr}{c_2} \right) - K_1 \left( \frac{pr}{c_1} \right) \frac{c_2}{c_1} \right)$$

$$\chi(r,p) = K_2 \left( \frac{pr}{c_2} \right) - \frac{c_2}{c_1^2} K_2 \left( \frac{pr}{c_1} \right)$$

Originated by $\bar{u}^2(x,p)$ stresses can be written as a convolution over $x$. 
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\[ \sigma_{ij}^2 = -\gamma S_{ij}^k \Theta(x, p)H_D(x)_k = -\gamma (S_{ij}^k \Theta(x, p)H_D(x)_k = \]

\[ = -\gamma(\Theta(x, p)H_D(x)\delta_{ij} - \int D(\Theta(y, p) - \Theta(x, p)) \Phi_{ij}^1(x - y, p) dV(y) - \]

\[ - \Theta(x, p) V \cdot P \int_S n_k(y) S_{ij}^k(x - y, p) dS(y)) \]

\[ \bar{S}_{ij}^k(x, p) = \lambda U_{mk} \cdot m \delta_{ij} + \mu \left( \bar{U}_{ik} \cdot j + \bar{U}_{jk} \cdot i \right) \]

There is the expression for the bounding stresses from here

\[ \bar{\sigma}_{ij}^2(x, p) = \gamma [ \int D(\Phi_{ij}^1(x - y, p)(\Theta(y, p) - \Theta(x, p))dV(y) + \]

\[ + c^2 \Theta(x, p) n_i n_j - 0.5(1 + 2c^2 \delta_{ij}) \Theta(x, p) + \]

\[ + \Theta(x, p) V \cdot P \int_S \bar{S}_{ij}^k(x - y, p)n_k(y)dS(y)) + \]

\[ + \int D(\Theta(y, p) - \Theta(x, p))\bar{S}_{ij}^k(x - y)dV(y)] \]

\[ \Phi_{ij}^1(x, p) = \bar{S}_{ij}^k(x, p) \]

The density \( \rho \) is the solution of singular BIE:

\[ 0.5 \bar{\sigma}_i(x, p) + \int S_{ik} (x - y, p, n(y)) \bar{\sigma}_k(y, p) dS(y) = \]

\[ = -\bar{\sigma}_{ij}^2(x, p) n_j(x) + \bar{f}_i(x, p) \]

\[ \bar{S}_{ik} (x, p, n) = \bar{S}_{ij}^k(x, p) n_j \]

The tensor \( \bar{\sigma}_{ij}^2 \) is defined from Eq. (7) after solving of Eq. (3) for \( x \in S \), then BIE (8) are solved numerically. Formulae (4)-(6) enable to find the displacement of medium.

4 Numerical realisation of BEM

The numerical realisation of BEM are based on boundary contour fragmentation and their interpolation by cubic spline, the construction of the discrete analogue of BIE on the base of the polynomial interpolation of determined functions with using of Gauss formulae for integrating and solving of corresponding system of linear equations, numerical inverse Laplace transformation to original time-space. Applied program in FORTRAN for IBM PC, solving external problems of uncoupled thermoelastodynamics, has been produced. This algorithm was tested on the problem for circular cavity in a plane by action heat flow of next kind

\[ q(x, t) = -H(t), \quad \sigma_{ij}(x, t) n_j(x) = 0, \quad x \in S \]
$H(t)$ is Heaviside function, characteristic function of set $t > 0$.

Here the some results of computation for a plane with arched form cavity (Fig.1) are considered for impulsive heat flow at the boundary:

$$q(x,t) = \begin{cases} 0 & t < 0 \\ t H(t) H(1-t) - (t-2) H(t-1) H(2-t) & t > 0 \end{cases}$$

$\sigma_{ij}(x,t)n_j(x) = 0 , \quad x \in S$

The height and the width of the arched hole are equal 2. The top part of hole boundary is the half-circle of radius $R=1$, bottom part is rectangular 2x1 with rounded corners with the radius 0.1. The boundary of the hole is divided to $N=36$ elements, at corners the elements have the smaller size. The heaviest size of a boundary element is equal 0.313, least this 0.031. To the hole the system of co-ordinates is connected, the axis $Oy$ coincides an axis of symmetry, and axis $Ox$ passes by the bottom part of boundary.

Hypothetical thermoelastic medium was used for the illustration of the opportunities of the BEM with next dimensionless parameters: $\rho = 1$, $v = 0.25$, $c_1 = 1$, $\gamma = 1$, $k = 1$.

At the figure 2 a,b the temperature $\theta(x, t)$ are plotted at various points of hole boundary for the problem (10) and (11) consequently: 1 - (0.0;0.0), 2 - (0.9;0.0), 3 - (0.9588;0.0191), 4 - (1.0;0.10), 5 - (1.0;1.0), 6 - (0.5878;1.8090), 7 - (0.0;2.0). Highest temperature is observed at the plane basis of the hole, i.e. in a point 1 with co-ordinates (0,0). During the time the temperature is increasing in first case and it has impulsive form in second case.

At the figure 3 a,b the dependence of stresses on time is represented for problems (10) and (11), where the numbering curve corresponds to numbering points laying on the boundary hole. But here the normal tangential stresses the $\sigma_{ij} \tau_j$ ( $\tau$ is unit vector perpendicular to $n$) are smallest at point 1 for both cases.
On the Fig.4 the stresses $\sigma_y \tau_j$ at other points for second problem are presented. Their increasing for great $t (t > 6)$ is connected with unstability of inverse Laplace transformation.

Figure 3: Normal tangential stresses on the basis and top of the hole.

Figure 4: Normal tangential stresses on the chosen points.
5 Boundary Integral Equations in original space

One of the main problems of BEM with using of Laplace transformation of time \( t \) is the great differences between arguments of kernels in BIE for transformant of temperature \( \theta(x,t) \) and displacements \( u(x,t) \) when the calculations are produced for natural thermoelastic media. If the sequence of Laplace parameters \( \{p_k\} \) is suitable for solving BIE (3), it is unsuitable for BIE (8) and vice versa. The cause of this is the behaviours of Macdonald functions \( K_n(z) \), which have singularity by \( z=0 \) and are decreasing exponentially by large \( z \). Because solving BIE for all and great kernel arguments is unstable. Also the choice of \( \{p_k\} \) by numerical calculation for inverse transformation is a complex problem. The problem of instability of inverse Laplace transformation is well known. Other difficulty for external problems is infinity of a domain of integration and necessity to calculate transformant of temperature inside such domain for every \( p^k \).

To avoid these problems, BIE may be constructed in the original space \( R^2 \times t \). We use for this goal the Green function of Eq.(2) corresponding \( Q=\delta(x)\delta(t) \) and Green tensor \( U_{ij}(x,t) \) which are determined by expressions

\[
4\pi U(x,t) = \frac{H(t)}{k^3 t} \exp(-r^2/4k^2 t) \\
2\pi U_{ij}(x,t) = (2r_i r_j - \delta_{ij}) \frac{t^2}{r^2} \left[ \frac{c_1 H(c_1 t - r)}{\sqrt{c_1^2 t^2 - r^2}} - \frac{c_2 H(c_2 t - r)}{\sqrt{c_2^2 t^2 - r^2}} \right] + \\
\left[ (\delta_{ij} - r_i r_j) \frac{H(c_1 t - r)}{c_1 \sqrt{c_1^2 t^2 - r^2}} + r_i r_j \frac{H(c_2 t - r)}{c_2 \sqrt{c_2^2 t^2 - r^2}} \right]
\]

Here \( r = \|x\|, r_i = x_i/r \). The investigation of asymptotic properties of this tensor has shown that it has not singularities by fixed \( t>0, r\to0 \), and it behaves like \( 1/t \) by \( t\to0 \). It has weak singularities at the wave fronts \( r = c_i t \).

It’s easy to prove from Eq.(3) that \( \theta(x,t) \) can be represented in the form:

\[
H_D(x)H(t)\theta(x,t) = \int_S dS(y) \int_0^t d\tau \frac{\partial U(x-y,t-\tau)}{\partial n(y)} \theta(y,\tau) d\tau - \\
- \int_S dS(y) \int_0^t d\tau q(y,\tau) U(x-y,t-\tau) d\tau
\]

For \( x\in S, t>0 \) this formula is the BIE for definition of bounding temperature. After determining \( \theta(x,t) \) at the boundary it can be defined inside medium.

For determining of a displacement it’s convenient to introduce also the next fundamental tensors.
The variable under a sign of convolution (*) means that this convolution is taken over this variable. If such symbol is absent the convolution is taken here over \((x,t)\).

The anti derivative \(V_y\) has logarithmically singularity by \(r = 0\) and the finite gaps at the fronts:

\[
2\pi V_y(x,t) = \delta_{ij} \sum_{k=1}^{2} \frac{H(c_k t - r)}{2c_k^2} \ln \left( \frac{c_k t + \sqrt{c_k^2 t^2 - r^2}}{r} \right) - \frac{(2r, t, j - \delta_{ij})}{r^2} \sum_{k=1}^{2} (-1)^k H(c_k t - r) \frac{t}{c_k} \sqrt{c_k^2 t^2 - r^2}
\]

It has been shown in a paper \(^9\) that

\[
W_{ij} = T_{ij}^S + W_{ij}^d
\]

Here \(T_{ij}^S(x,n)\) is the fundamental solution of elastostatic equations, like \(T_{ij}\). (15) Its asymptotic properties are well known \(^10\). In particular \(T_{ij}^S(x,n)\) has strong singularity by \(x = 0\):

\[
T_{ij}^S(x,n) \sim K_{ij} ||x||, \quad ||x|| \to 0,
\]

where \(K_{ij}\) are limited and it satisfies to Gauss formula

\[
\int_{S} T_{ij}^S(x - y, n(y)) dS(y) = H_D(x).
\]

Dynamic part \(W_{ij}^d(x,t,n)\) has only weak singularities at wave fronts. With use of these tensors the analogue of Somigliana’s formula for solutions of elastodynamics equations can be written in the form \(^9\):

\[
H_D(x) H(t) u_i(x,t) = U_{ij} * G_j + \int_{S} dS(y) \int_{0}^{t} U_{ij}(x - y, \tau) p_j(y, \tau) d\tau + \int_{S} dS(y) \int_{0}^{t} W_{ij}^d(x - y, \tau, n(y)) u_j, t(y, \tau - \tau) d\tau + \int_{S} dS(y) T_{ij}^S(x - y, n(y)) u_j(y, t) dS(y)
\]

For \(x \in S\) the Eq. (14) are the BIE for determination of \(u_i(x,t)\). Here all integrals except last one are regular. The last integral is singular and exists in sense of principal value. The expression of all these tensors, suitable for calculations, see in the paper \(^11\).

The first convolution in the relation (16) can be written in the form
\[ U_{ij} \cdot G_j = -\gamma U_{ij} \cdot (H(t)H_D(x)\theta(x,t)), \quad j = -\gamma U_{ij} \cdot H(t)H_D(x)\theta(x,t) \]  

(17)

where

\[ 2\pi c_1^2 U_{ij} = \frac{\partial}{\partial x_i} \frac{H(c_1 t - r)}{\sqrt{c_1^2 t^2 - r^2}} = \frac{H(c_1 t - r)x_i}{\sqrt{(c_1^2 t^2 - r^2)^3}} - \frac{\delta(c_1 t - r)\mathbf{r}_i}{\sqrt{c_1^2 t^2 - r^2}}, \quad r = \|\mathbf{r}\| \]

Let \( S_k^-(x,t) = \{ y \in D : \| x - y \| < c_k t \} \), \( S_k(x,t) = \{ y \in S : \| x - y \| < c_k t \} \).

Here \( f(x, t)\delta(c_1 t - r) \) is the simple layer with the density \( f(x, t) \). Then the expression (15) can be written as

\[ U_{ij} \cdot G_j = \frac{\gamma}{2\pi c_1^2} \left( \int_0^t \int_{S_1^-} \frac{r_i \theta(y, t - \tau)}{\sqrt{(c_1 t^2 - r^2)^3}} \, dS(y) - \int_0^t \int_{S_1^-} \frac{r_i \theta(y, t - \tau)}{\sqrt{(c_1 t^2 - r^2)^3}} \, dV(y) \right), \quad r = \| x - y \|, \quad r_i = \partial r / \partial y_i. \]

Here all integrals are regular with weak singularities at wave front of kernels \( r = c_1 \tau \). It’s easy to see, the domain of integration \( S_1^- (x, t) \) is always limited.

Numerical solving Eq. (16) for \( x \in S \) step by step on time \( t \) gives the temperature at a boundary. Then we find \( \theta(x, t) \) inside \( S_1^- (x, t) \) for \( \forall x \in S \) to define \( u(x, t) \) from Eq. (16) at \( S \). After these operations we can determine the state of medium in every chosen points.

REFERENCES


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