A boundary integral technique for two-dimensional potential flows

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Abstract

A boundary integral technique is developed for the steady two-dimensional, irrotational, incompressible fluid flow in complex geometries. Using the Dirichlet boundary problem for an analytical function on the upper half plane gives rise to integral-differential equations on the boundary of the solution domain in the physical plane. The boundary integral equation is discretized and an iterative procedure is developed to solve the resulting algebraic equations. This method is different to the conventional boundary element method, which requires the inversion of a large system of linear algebraic equations, in that it requires much less computer memory and computing time. Two examples of the technique are presented.

1 Introduction

The mathematical formulation of many problems in engineering and physics naturally gives rise to a boundary value problem which consists of a governing partial differential equation in conjunction with a set of prescribed boundary conditions. The boundary element method has the great advantage in that the governing equations have only to be solved on the boundary of the solution domain. The boundary element method, based upon the two-dimensional form of Green's Integral Formula, has been extensively used to solve Laplace type equations by Jaswon and Symm\(^1\). Whilst, Ingham and Kelmanson\(^2\) extended the method to solve the biharmonic equation. A very detailed description of the boundary element method may be found in Brebbia and Walker\(^3\) and the vast range of applications of this method may
be found in a large variety of publications in journals and conferences.

In this paper a new boundary integral method for the solution of steady, two-dimensional potential flows in complex geometries is developed. The solution domain in the \( \phi-\psi \) plane is transformed into the upper half plane of an auxiliary \( t \)-plane and then the solution of the potential flow problem is obtained on this upper-half of the \( t \)-plane for the Dirichlet problem. By transforming back into the physical plane a system of boundary integral and differential equations along the boundary in the physical plane are derived. The numerical discretisation of the boundary integral equations are performed by a linear interpolation and a numerical iterative procedure is proposed in order to solve the resulting nonlinear, integral-differential equations and this produces the solution on the boundaries of the physical plane.

2 Mathematical method

Suppose that \( \Omega(t) = f(t) + ig(t) \) is an analytical function on the upper-half plane, where \( t = \zeta + i\eta \) is a complex variable. When the real part \( f(t) \), or the imaginary part \( g(t) \), of the analytical function \( \Omega(t) \) is known on the real axis of the \( t \)-plane then this analytical function is a solution of the Dirichlet boundary problem\(^4\). When the imaginary part of \( \Omega(t) \) is known, the Dirichlet problem states: to find an analytical function \( \Omega(t) \) on the upper-half of the \( t \)-plane when the function \( g(\zeta) \) is known on the real axis and it satisfies the Holder condition at \( \zeta = \zeta_0 \) on the axis, i.e. for all values of \( \zeta \) on the real axis which are sufficiently close to \( \zeta_0 \), then

\[
| g(\zeta) - g(\zeta_0) | \leq A | \zeta - \zeta_0 |^\mu \quad \text{and} \quad 0 < A < \infty, \quad 0 < \mu < 1 \quad (1)
\]

The solution of this boundary problem is given by the Schwarz formula\(^4\), namely

\[
\Omega(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\zeta)}{\zeta - t} d\zeta + \Omega_\infty \quad (2)
\]

where \( \Omega_\infty \) is the value of the function \( \Omega(t) \) at \( t = \infty \).

To use equation (2), to obtain the solution of the steady, two-dimensional, incompressible potential flow problem, we choose \( z = x + iy \) as the coordinate in the physical plane and \( W = \phi + i\psi \) as the complex velocity potential which given by

\[
dW/dz = u e^{-i\theta} \quad (3)
\]

where \( u \) is the fluid speed and \( \theta \) is the angle that the fluid velocity vector makes with the positive \( x \) axis. In the \( W \)-plane the fluid velocity \( u \) and the
angle $\theta$ are functions of $W$, and equation (3) may be written in the form
\[ dW/dz = u(W)e^{-i\theta(W)} \]  
(4)

In the velocity potential plane, $W$, the solution domain has a very simple geometry, e.g. the whole plane, the half plane, a strip or a plane with cuts, because on the boundaries of the physical plane the stream function takes constant values. For such simple solution domains in the $W$-plane it is easy to find an analytical function which conformally maps the solution domain in the $W$-plane onto the upper-half of an auxiliary $t$-plane. Suppose this conformal mapping function is known and is written in the form
\[ W(t) = \phi(t) + i\psi(t), \quad \text{or} \quad t(W) = \zeta(\phi, \psi) + i\eta(\phi, \psi) \]  
(5)

then on the $t$-plane, where $t = \zeta + i\eta$, equation (4) becomes
\[ dW/dz = u(t)e^{-i\theta(t)} \]  
(6)

The logarithm of the complex velocity, $\Omega$, may now be introduced as follows:
\[ \Omega(t) = \ln\left(\frac{1}{U} \frac{dW}{dz}\right) = \ln\left[\frac{u(t)}{U}e^{-i\theta(t)}\right] = \tau(t) - i\theta(t) \]  
(7)

where $U$ is the velocity at infinity and $\tau(t) = \ln\frac{u(t)}{U}$ and $\Omega(t)$, $\tau(t)$ and $\theta(t)$ are analytical functions on the upper half of the $t$-plane. At the infinite point we have $\tau(\infty) = 0$, and if $\theta(\infty) = 0$ then $\Omega_{\infty} = 0$.

If on the real axis of the $t$-plane the angle $\theta(\zeta)$ is a known function, namely $\theta(\zeta)$ is given, then by applying equation (2) and the condition $\Omega_{\infty} = 0$, the solution for $\Omega(t)$ may be written in the form
\[ \Omega(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(\zeta)}{\zeta - t} d\zeta \]  
(8)

On letting $t$ approach the point $\zeta_0$ on the real axis from the upper half plane and taking the Cauchy principle value we obtain
\[ \Omega(\zeta_0) = \tau(\zeta_0) - i\theta(\zeta_0) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(\zeta)}{\zeta - \zeta_0} d\zeta - i\theta(\zeta_0) \]  
(9)

On comparing the real and imaginary parts of both sides of equation (9), we find that the fluid velocity on the real axis $\zeta$ is given by
\[ \tau(\zeta_0) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\theta(\zeta)}{\zeta - \zeta_0} d\zeta \]  
(10)

On the boundary of the physical plane, $\Gamma$, the fluid velocity is in the tangential direction to the boundary and therefore the function $\theta(x, y)$, or
\( \theta(s) \), is a known function in the \( x-y \) Cartesian coordinate system, or the arc coordinate, \( s \), along the boundary in the physical plane. However, on the boundary of the \( W \)-plane, the function \( \theta(\phi, \psi) \), where \( \psi \) is constant, is an unknown function because we do not know the functions \( \phi(x, y) \) or \( \phi(s) \). Therefore in equation (1) the quantity \( \theta(\eta) \) is an unknown function of \( \eta \). In order to overcome this difficulty we take the arc coordinate, \( s \), as a variable. Thus all the quantities are now only functions of \( s \) and we also have

\[
d\zeta = \frac{d\zeta}{d\phi} \frac{d\phi}{ds} ds
\]

and on the boundary of the physical plane the velocity potential is connected to the fluid velocity by the differential equation

\[
\frac{d\phi(s)}{ds} = u(s)
\]

Substituting equations (11) and (12) into expression (10) we obtain

\[
\tau(s_0) = \ln \frac{u(s_0)}{U} = -\frac{1}{\pi} \int_\Gamma \frac{\theta(s)}{\zeta(s) - \zeta(s_0)} \frac{d\zeta}{d\phi} u(s) ds
\]

where the integration is performed on the boundary, \( \Gamma \), in the physical plane, on which the only unknown functions are \( \phi(s) \) and \( u(s) \), \( \theta(s) \) is a known function of \( s \) and \( \zeta(s) \) is determined from equation (5). Therefore, equation (13) is a boundary integral equation for the unknown functions \( u(s) \) and \( \phi(s) \). By solving the integral-differential equations (12) and (13), in which \( \zeta \) is connected to \( \phi \) by equation (5), the solution of the potential flow can be obtained by the iterative scheme as developed in Refs [5,6].

3 Numerical discretization

In order to perform numerical calculation we divide the real axis into segments, \( \Gamma_i, i=1, ..., N \). Correspondingly equation (10) becomes

\[
\tau(\zeta_k) = \ln \frac{u(\zeta_k)}{U} = -\sum_{i=1}^{N-1} \frac{1}{\pi} \int_{\zeta_i}^{\zeta_{i+1}} \frac{\theta(\zeta)}{\zeta - \zeta_k} d\zeta
\]

Over each interval we approximate \( \theta(\zeta) \) in equation (10) by linear functions of \( \zeta \), namely,

\[
\theta(\zeta) = \frac{\zeta_{i+1} - \zeta}{\zeta_{i+1} - \zeta_i} \theta_i + \frac{\zeta - \zeta_i}{\zeta_{i+1} - \zeta_i} \theta_{i+1}
\]

When \( \zeta_k \neq \zeta_i, \zeta_{i+1} \) we have

\[
\int_{\zeta_i}^{\zeta_{i+1}} \frac{\theta(\zeta)}{\zeta - \zeta_k} d\zeta = \theta_i \left[ \frac{\zeta_{i+1} - \zeta_k}{\zeta_{i+1} - \zeta_i} \ln \left( \frac{\zeta_{i+1} - \zeta_k}{\zeta_i - \zeta_k} \right) - 1 \right]
\]

\[
+ \theta_{i+1} \left[ \frac{\zeta_k - \zeta_i}{\zeta_{i+1} - \zeta_i} \ln \left( \frac{\zeta_{i+1} - \zeta_k}{\zeta_i - \zeta_k} \right) + 1 \right]
\]
whilst when $\zeta_k = \zeta_1$ we have

$$
\int_{\zeta_{k-1}}^{\zeta_{k+1}} \frac{\theta(\zeta)}{\zeta - \zeta_k} d\zeta = \theta_k \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_k - \zeta_{k-1}} + \theta_{k+1} - \theta_{k-1}
$$

(17)

4 Application of the method

In the following we illustrate, by the use of two examples how the theory developed above may be used to obtain the solution of two potential flow problems by developing a numerical iterative scheme for the integral-differential equation (12) and (13).

4.1 Flow in a semi-infinite domain

Figure 1a: Potential flow in a semi-infinite domain.

Figure 1b: The $W$-plane ($t$-plane).

Figure 1 shows the solution domain for the fluid flow over an arbitrarily shaped hump in the upper half plane. This potential flow has a uniform fluid velocity $U$ far upstream, $AF$, of the hump and the boundaries $AB$ and
CD are taken to be horizontal and the shape of BC, the two humps, is given by the function \( y = f(x) \) and the tangential angle \( \theta(s) \) on the boundary BC is specified by \( \frac{dy}{dx} \). We assume that the stream function on the boundary ABCD is zero and in the \( W \)-plane the solution domain is the upper half plane, see figure 1b. In this simple case the mapping function (5) takes a very simple form, namely \( t = W \), i.e. \( \zeta = \phi \) and \( \eta = \psi \). Therefore the equation (13) takes the form:

\[
\ln \frac{u(s_0)}{U} = -\frac{1}{\pi} \int \frac{\theta(s)}{\zeta(s) - \zeta(s_0)} u(s) ds
\]

This equation indicates that at any point \( s_0 \) on the boundary ABCD, the fluid velocity \( u(s_0) \) may be determined by performing a boundary integration and the nodes may be distributed in the same manner as when using the boundary element method. The equations (18) and (13) are solved by using the following iterative scheme:

(a) Distribute the grid points on ABCD, then \( \theta(s) \) is given at each point through the specified geometry of ABCD.

(b) Assume an initial fluid velocity distribution \( u^n(s) = U \), say, on the boundary ABCD, where the upper suffix represents the number of iterations. Initially \( n= 0 \)

(c) The function, \( \phi \), on ABCD is calculated by integrating equation (13), namely,

\[
\phi^n(s) - \phi^n(s_A) = \int_{s_A}^{s} u^n(s) ds.
\]

(d) On substituting these values of \( \theta(s), u^n(s) \) and \( \phi^n(s) \) into the right hand side of equation (18), a new fluid velocity distribution \( u^{n+1}(s) \) is obtained.

(e) Repeat the steps (c) and (d) until the fluid velocity, \( u^n(s) \), and the velocity potential \( \phi^n(s) \) are convergent. In all the calculations presented in this paper only 5 or 6 iterations are necessary and the tolerances at all the computational points are such that

\[
| u^{n+1}(s) - u^n(s) | \leq 10^{-5} \quad \text{and} \quad | \phi^{n+1}(s) - \phi^n(s) | \leq 10^{-5}
\]

are satisfied.

For illustrative purpose, we consider a curvilinear boundary BC with two-humps which are given by \( y = \frac{1}{4} [\cos(x-1)\pi -1] \) in the range \( x \in (-2.0, 2.0) \), and the upstream point A is taken to be located at (-6.0, 0.0) and the downstream point D is at (6.0, 0.0). Figure 2 shows the resulting
streamlines and potential lines.

![Streamlines and Potential Lines](image)

Figure 2: The streamlines and potential lines in a semi-infinite domain.

### 4.2 Flow in a two-dimensional arbitrarily shaped channel with a bifurcation

Figure 3a schematically shows a two-dimensional channel with a bifurcation where far upstream the channel is bounded by two parallel straight lines and the width of the channel is $H$ and $\theta_1(s)$, $\theta_2(s)$, $\theta_3(s)$ and $\theta_4(s)$ are the angles that the boundaries of the channels AB, CB, CD, and AD, respectively, make with the horizontal. A potential flow goes from the upstream channel into the two downstream channels and the point C is the stagnation point. Upstream there is a uniform flow which has a speed $U$ and the flux of fluid is $q = UH$. In the downstream channels, say channel 1, the width of the channel is $H_1$, the fluid speed is $U_1$ and the flux of fluid is $q_1 = U_1H_1$, whereas in channel 2 the width of the channel is $H_2$, the fluid speed is $U_2$ and the flux of fluid is $q_2 = U_2H_2$. Clearly we must have $q = q_1 + q_2$. In this case the fluxes $q_1$ and $q_2$ are determined by the position of the stagnation point C, or the position of the stagnation point C is determined by the fluxes $q_1$ and $q_2$. In our calculations we fix the position of the stagnation point at the point where the geometry is most convex in the solution domain, see figure 3a. In figure 3b the solution domain in the $W$-plane is a strip with a cut and the $W$-plane may be mapped onto the upper-half of the $t$-plane by the use of the Schwarz-Christoffel transformation, namely, the mapping function (5) takes following form

$$ W = -\frac{q_1}{\pi} \ln(t + 1) - \frac{q_2}{\pi} \ln(t - 1) + c $$

(21)
where \( c \) is a constant and without any loss in generality we assume at the stagnation point that the potential function takes the value zero. Thus expression (21) reduces to

\[
W = -\frac{q_1}{\pi} \ln \frac{t + 1}{C_s + 1}
\]  

(22)

which transforms the points A, B, C and D into the infinite point, -1, \( C_s \) and + 1, respectively, see figure 3c, and \( C_s \) is determined by

\[
C_s = \frac{q_1 - q_2}{q_1 + q_2}
\]  

(23)
Applying equation (10) on the real axis of the \( t \)-plane, we obtain the expression for the fluid velocity on the real axis of the \( t \)-plane, namely

\[
\ln \frac{u(\zeta_0)}{U} = -\frac{1}{\pi} \int_{-\infty}^{-1} \frac{\theta_1(\zeta)}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_{-1}^{C_0} \frac{\theta_2(\zeta)}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_{C_0}^{+1} \frac{\theta_3(\zeta)}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_{+1}^{+\infty} \frac{\theta_4(\zeta)}{\zeta - \zeta_0} d\zeta
\]  

(24)

By use of equations (11), (12) and (21), equation (24) may be transformed into boundary integration, i.e. equation (13) or (24) takes the form

\[
\ln \frac{u(s_0)}{U} = -\frac{1}{\pi} \int_{\Gamma_1} \frac{\theta_1(s)}{\zeta - \zeta_0} \frac{d\zeta}{d\phi} u(s) ds - \frac{1}{\pi} \int_{\Gamma_2} \frac{\theta_2(s)}{\zeta - \zeta_0} \frac{d\zeta}{d\phi} u(s) ds - \frac{1}{\pi} \int_{\Gamma_3} \frac{\theta_3(s)}{\zeta - \zeta_0} \frac{d\zeta}{d\phi} u(s) ds - \frac{1}{\pi} \int_{\Gamma_4} \frac{\theta_4(s)}{\zeta - \zeta_0} \frac{d\zeta}{d\phi} u(s) ds
\]  

(25)

The equations (19), (25) and (22) for \( u(s) \), \( \theta(s) \) and \( \zeta(s) \), respectively, are solved using the same iterative scheme as developed in example 1 and during the iteration the fluxes of fluid \( q_1 \) and \( q_2 \) are calculated by use of the fluid speeds downstream of the channels 1 and 2.
Figure 4 shows the streamlines and the potential lines for an asymmetrical bifurcation where the boundary CB consists of a curve described by the function \( x = y^2 \) and the straight line which makes an angle of \(-13.5^\circ\) with the positive x axis, the boundary CD consists of a curve described by the function \( x = 3y^3 \) and the straight line which makes an angle of \(35.5^\circ\) with the positive x axis, the boundaries AB and AD consist of two straight lines and a cubic curve, respectively. In order to illustrate the results, in all the calculations the upstream point A was located at \( x = -10.0 \) and the downstream points B and D were at \( x = 10.0 \) but the effect of moving these locations further from the origin produces results which are almost graphically indistinguishable from those shown in figure 4.

5 Conclusion

A new boundary element method has been developed for two-dimensional, steady potential flows based on the Dirichlet boundary problem for analytical functions. The resulting non-linear boundary integral equations are solved by using an iterative method which has a fast rate convergence. Because there is no need for the inversion of a large system of algebraic equations, and the iterative procedure has a very fast rate of convergence, this technique requires much less computer memory and smaller computing times than does the boundary element method and the Schwarz-Christoffel technique.

References