A Boundary Element Method for analysis of bone remodelling

M. Martinez, M.H. Aliabadi, H. Power

Wessex Institute of Technology, Ashurst Lodge, Ashurst, Southampton SO40 7AA, UK

Abstract

A boundary element formulation is presented for analyzing surface bone remodelling. The formulation is based on sensitivity analysis and utilises design parameters which are related to the shape of the bone. Application of the method to analysis of remodelling of the diaphyseal region of a long bone subjected to an internal pressure is presented. Several examples for circular and elliptical bone cross sections are presented to demonstrate the robustness of the proposed method.

1 Introduction

The human body is a very complex structure in which the skeleton plays an important role. The skeleton is the frame that supports and protects various vital organs and tissues, moreover, without a skeleton it would be impossible to walk. In all there are 206 separate bones in an adult body. At the macroscopic level there are two major kinds of bone tissue, called compact or cortical bone which is the dense type of bone and cancellous bone which is a more porous or spongy type of bone. The diagram of a human femur is shown in fig 1. This work is concerned in presenting results obtained on cortical bone remodelling.

In biomechanics, bone remodelling is known as the process by which bone adapts its shape to the mechanical environment. Cowin and Van Buskirk [2] have presented a theory of surface bone remodelling which can be used to predict the changes of the external shape of a bone. To simulate the remodelling process using computational methods there have been several publications based on the Finite Element Method (FEM) (see, for example [5]). In FEM it is necessary to discretise the domain of the bone and to update the mesh after each change to the boundary shape. The Boundary Element Method (BEM) is an alternative tool for shape prediction problems yielding accurate boundary solutions and requiring no domain discretisation. The BEM approach was applied by Sadegh et al. [6] to the study of bone remodelling along implant interface. In their analysis the problem was considered as a moving boundary problem, and the surface velocity was calculated using Cowin and Van Buskirk [1] remodelling law. The boundary was moved at each time step using the BEM analysis.

In this paper a boundary element formulation is presented to predict the movement of periosteal and endosteal surfaces to the diaphyseal region of a long bone. Usually, the analysis of surface remodelling problems as a moving boundary problem requires many iterations to determine the remodelling equilibrium. In order to reduce this effort the proposed method combines the boundary element analysis, sensitivities analysis, error analysis and optimisation method. To demonstrate the accuracy of the new formulation, bone remodelling problems of the diaphyseal region of a long bone with circular or elliptical cross section subjected to an internal pressure is analysed.
2 Surface Bone Remodelling

Surface remodelling refers to the resorption or deposition of bone material on the external surface of the bone. In Fig 2. Cross sections of two rabbit tibia are shown in order to illustrate the remodelling process. The cross hatched is to indicate the bone deposition during of period of study. On the left the cross area is assumed to be associated with normal growth. On the right the tibia has been subjected to hyperphysiological bending. This study was done by Liskova and Hert[9] showing that intermittent bending applied to the rabbit tibia can cause the periosteal surface to move out. Surface bone remodelling assumes that the bone tissue can be modelled as a linear elastic body whose free surfaces move according to an additional constitutive equation relating to the changes from a mechanical state to the remodelling equilibrium. Cowin and Van Buskirk [1] have proposed external surface remodelling equations as:

\[ U(X) = C_{ij}(X) \left[ \varepsilon_{ij}(X) - \varepsilon_{ij}^0(X) \right] \]  

(1)

where \( C_{ij}(X) \) are the surface strain remodelling rate constants, \( \varepsilon_{ij}(X) \) is the actual strain tensor at point \( X \), \( \varepsilon_{ij}^0(X) \) is the remodelling equilibrium strain tensor at point \( X \) and \( U \) is the velocity of the remodelling surface at point \( X \). If the strain state at \( X \), \( \varepsilon_{ij}(X) \), is equal to the reference strain state at \( X \), \( \varepsilon_{ij}^0(X) \), then the velocity of the surface is zero, \( U(X) = 0 \), and no remodelling occurs.
The right hand side of equation (1) can be expressed in terms of stress instead of strain as

\[ U(X) = B_{ij}(X) [\sigma_{ij}(X) - \sigma_{ij}^0(X)] \]

(2)

where \( B_{ij}(X) \) are the surface stress remodelling rate coefficients at the point \( X \), \( \sigma_{ij}(X) \) is the actual strain tensor at point \( X \) and \( \sigma_{ij}^0(X) \) is the remodelling equilibrium stress at \( X \). The relation between equations (1) and (2) is implied by Hooke's law.

The expression in equation (2) for the surface remodelling has the advantage, over the expression (1) for the same quantity, of directly depending on the load and the geometric properties and only indirectly on the values of the bone’s elastic constants (Cowin et al., [4]).

In this paper it is assumed that responses of the actual bone, due to an applied field, are known only at various selected points on the boundary of the bone (called sensor points), which can be situated anywhere on the boundary of the bone.

In this work, the known stresses at sensor node \( X^n \) from the final bone shape are denoted by \( \sigma_{ij}^n(X^n) \), whilst the computational stresses, from an assumed bone shape are denoted by \( \sigma_{ij}^{BEM}(X^n) \). This allows the error at sensor node \( X^n \) to be defined by

\[ S(X^n) = \sigma_{ij}^n(X^n) - \sigma_{ij}^{BEM}(X^n) \]

(3)

for values measured at sensor node \( X^n \). The error at all \( N \) sensor points on the boundary of the bone can be quantified into an error function, \( F(Z) \). The error function used here, is defined by

\[ F(Z) = \sqrt{\sum_{n=1}^{N} \sum_{i,j=1}^{2} (S(X^n))^2} \]

(4)

where \( Z \) is the vector of design variables used to define the assumed bone shape.

This error function will then be minimised with an optimisation technique in order to obtain the final bone shape corresponding to a zero surface velocity in equation (2).

3 Sensitivity

If bone remodelling is seen as structural analysis then it is possible to assume that a boundary value problem can be formulated in the same manner as a boundary problem...
Boundary Elements

in linear elastostatic, but it is necessary to specify the boundary conditions for a specific time period. As the body evolves to a new shape, the stress and strain must be varying quasistatically. At any instant the body must behave exactly as an elastic body, but moving boundaries cause local stress and strain to redistribute themselves slowly with time (Cowin et al., [3]).

The objective of the formulation is to predict the final bone shape when the remodelling equilibrium stresses have been achieved on the boundary of the bone. This is done by defining a set of variables which describes the shape of the bone.

![Figure 3: Design variables for sensitivity analysis](image)

In the Fig.3 several examples of design variables \((z_m)\) used to describes both a circular transversal section and an elliptical transversal section of a bone are presented. If the bone transversal section is assumed to be a circular with predefined central location the design variables can be defined by (a) both radii \(R_1\) and \(R_2\), (b) the difference of the radii \(C\) with \(R_2\), (c) the difference of radii \(C\) with \(R_1\), (d) the difference of the radii \(C\) with \(R_1\). However, if the bone transversal section is assumed to be elliptical it is necessary to define two design variables (e) differences between both elliptical minor and major axes, \(C_1\) and \(C_2\).

The method is as follows. The initial boundary of the bone is divided into \(N\) boundary elements. A set of points called sensor points are chosen on the boundary. It is assumed that stress values from the actual equilibrium are known at the sensor points. Next, using BEM analysis, stress values are obtained at the sensor points for the initial shape of the bone. Generally, stress values from BEM and stress values from the actual equilibrium are different leading to the definition of an error function \(F(Z)\), given by equation (4), where \(Z = (z_1, z_2, ..., z_m)\) is a vector of design variables \(z_m\) and the error at sensor points is given equation (3). Then the gradient of \(F(Z)\) can be written as
\n\nwhere
\n\[ \n\n\n\n\text{for stress values evaluated at sensor points } x^n. \n\]

\section*{4 Numerical Implementation}

The numerical procedure used to evaluate the boundary values and their derivatives is as follows

\subsection*{4.1 Evaluation of the displacement and traction values}

The discretised form of the BEM equation can be written as
\n\n\text{After the application of the boundary conditions, equation (7) can be written as}
\n\[ \nAx = f \]
\nwhere, \( A \) is the coefficient matrix, \( x \) is the vector of unknown values and \( f \) is evaluated from application of the known boundary conditions.

\subsection*{4.2 Evaluation of the displacement and traction derivatives}

The displacement and traction derivatives are evaluated by direct differentiation of boundary integral equations (7), with respect to the design variable \( z_m \).

These equations contain displacement and traction derivatives \( u^n_j \) and \( t^n_j \), at the boundary nodes. Using the displacement and traction values computed from the first step a system of equations can be formed from the derivative equations (7) for each of the design variables \( z_m \). The final of equations obtained can be written in matrix form as
\n\n\text{which can be rewritten in terms of the unknown variables } x, \text{ as}
\n\[ \nA x_m = G^m \]
\nwhere \( A \) is the same matrix used in equation (8), \( G^m = f_m - A_m x \) and \( x_m \) is the vector of unknown displacement and traction derivatives.
4.3 Shape bone updating

From an initial design variable vector $Z^0$, the objective function $F$ is evaluated. Using the first order, non-linear, unconstrained Quasi-Newton (BFGS) optimisation algorithm ([7], [8]) a search direction $d^0$ is computed in terms of the gradient of the error function. From this search a general update is made to the design vector by

$$Z^1 = Z^0 + \alpha^0 d^0$$  \hspace{1cm} (10)

the line step parameter $\alpha^0$ is computed by a line search technique in order to obtain the minimum design point along the search direction $d^0$.

The numerical process above mentioned is continued until the tolerance ($e$) required in the error function is reached that is to say

$$F(Z^{k+1}) - F(Z^k) \leq e$$  \hspace{1cm} (11)

5 Test Examples

In this section several test examples are presented to demonstrate the efficiency of the formulation proposed. The problems considered here is the surface remodelling that would occur in the diaphysial region of a long bone if an internal pressure carried by the bone were suddenly raised to a new level and held there indefinitely.

![Figure 4: BEM model of the cross section of the bone](image)

The cross section of the diaphysial region is modelled as circular or elliptical. This model is divided into 16 quadratic elements that is to say 32 nodes which are all used as sensor points such as is shown in Fig .4.
The Young's modulus $E = 17.12$ MPa, a Poisson ratio $\nu = 0.28$ and a Shear modulus $G = 6690$ Mpa are used as material properties of bone. As the main objective was to know how the bone shape goes from an initial to a final equilibrium, a unit pressure was used in all cases, that is to say the boundary conditions used here were a unit pressure on the endosteal surface and the periosteal free of loading. As the initial shapes are different, this means that the initial stresses due to this pressure were different. In the cross-section circular examples the stress resulting from a bone shape with $R_1 = 5$ and $R_2 = 10$, are used as stress values on the all sensors points and at the same time corresponding to the remodelling equilibrium state assumed. Fig. 5 shows the numerical examples and results obtained. In the second column all initial bone shapes used to start the remodelling process are presented. The third column shows all final bone shapes, in other words, the shapes that bone reach in the remodelling equilibrium. The fourth and fifth columns, arranged below of surface movement subtitle, shows the initial bone dimensions used to start the remodelling process and the final bone dimensions reached in remodelling equilibrium. Finally, in the sixth column the number of iterations of each process are shown.

In case No 1, it can be observed that both internal and external radii grow during the

<table>
<thead>
<tr>
<th>Examples</th>
<th>Initial shape</th>
<th>Final shape</th>
<th>Surface movement</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$R_1 = 4$</td>
<td>$R_1 = 5$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$R_1 = 6$</td>
<td>$R_1 = 5$</td>
<td>71</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$R_1 = 5$</td>
<td>$R_1 = 5$</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$R_1 = 4$</td>
<td>$R_1 = 5$</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$R_1 = 5.5$</td>
<td>$R_1 = 5$</td>
<td>89</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$R_1 = 4.5$</td>
<td>$R_1 = 5$</td>
<td>61</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>$a = 1.0$</td>
<td>$a = 2.0$</td>
<td>39</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$a = 1.0$</td>
<td>$a = 0.5$</td>
<td>27</td>
</tr>
</tbody>
</table>
remodelling process, corresponding to bone material deposition on the endosteal surface and bone material resorption on the periosteal surface. In the second case, conversely to the first one, both radii decreases, that is to says from the point of view of the remodelling process in the endosteal surface was a resorption of bone material and the periosteal surface undergo a deposition. In the cases No 3 and No 4, there is only a endosteal surface movement or a periosteal surface movement, respectively. The cases five and six are very interesting due the fact that the bone surfaces have movements in opposite direction and in order to obtain the remodelling equilibrium state was necessary to use two design variables. Finally, the examples seven and eight were considered for testing the formulation developed with other different geometric to the circular. The examples with elliptical cross sections are similar to the cases 1 and 2. It can be observed that the cross-sectional area increases in the case seven and decreases in the case eight as the consequence of the movements of both semi-axes a and b.

6 Conclusion

A new method for simulating surface bone remodelling process was presented. It is shown that the boundary element method in conjunction with sentivities analysis is an excellent tool for obtaining the equilibrium bone shape after surface remodelling. Several, test examples were carried out, demonstrating that the method proposed is suitable for solving two-dimensional bone shape problems related to surface remodelling theory.

References