A smooth fundamental solution for 3D time domain BEM formulations

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Abstract

In this work a particular fundamental solution of the 3D Time Domain BEM formulation is proposed to give the desired generality, accuracy and stability. Based on this fundamental solution it will be possible, at a near future, to treat initial conditions, velocity and displacement, and to deal with integral representations for stresses and velocities with great stability and confidence.

1 Introduction

Since the first work on the 3D time domain BEM formulation, written by Karabalis and Beskos [1], the stress integral representation has not yet been successfully implemented. This lack in the literature is present until the work due to Coda and Venturini [2], where initial conditions have been consistently treated, as well as a good idea on the desired properties of an ideal fundamental solution has emerged.

The authors have written several other works where these fundamental solution characteristics are discussed [3,4,5], but, in spite of reporting several times the required features of the ideal fundamental solution, they have not yet achieved and implemented the final expressions.

In this work, the complete values of a fundamental solution, that is much more powerful than the recently one, published by Coda [6], and Coda and Venturini [4], are presented. These fundamental values are characterised by exhibiting a much smoother time behaviour than the earlier formula, what make possible to treat stress and velocity representations, as well as general initial conditions by means of constant time approximation and using simple schemes to perform numerically the spatial integrals. These advantages avoid the use of some difficult analytical procedures already employed in 2D formulation [7,8]. The fundamental values proposed here can be easily extended to 2D elastodynamics and to problems governed by the scalar wave equation.

2 Basic equations

The motion of elastic bodies is governed by the Navier-Cauchy equation, as follows,
where \( \mathbf{b}_i \) represents body forces, \( \rho \) is the media density, and \( C_1 \) and \( C_2 \) are the longitudinal and shear wave velocities, respectively.

The general fundamental solution of eqn (1) is due to Stokes \([9,10]\) and is referred to the following load function:

\[
\mathbf{b}_{ki}^* = f(\tau) \delta(q-s) \delta_{ki}
\]

in which \( s \) and \( q \) are the load and field points respectively, and \( f(\tau) \) gives the load time behaviour.

Using the general solution, associated to the load function exhibited in eqn (2), together with the Graffi's reciprocity theorem \([11]\), one can write the final displacement integral representation in the same form several times shown in the literature. For a body of domain \( \Omega \) and boundary \( \Gamma \), submitted to a loading process, that began at a time \( \tau = 0 \) and is extended to a time \( t \), the displacement integral representation is reduced, as follows,

\[
C_{ki}(Q,s) \int_0^t u_i(s,\tau)f(t-\tau)d\tau = \int_0^t \int_{\Gamma} u_{ki}^*(Q,t-\tau; s/f)p_i(Q,\tau)d\Gamma d\tau + \\
-\int_0^t \int_{\Gamma} u_i(Q,\tau)p_{ki}^*(Q,t-\tau; s/f)d\Gamma d\tau + \int_{\Omega} u_{ki}^*(q,t-\tau; s/f)b_i(q,\tau)d\Omega d\tau + \\
+\rho \int_{\Omega} [u_{ki}^*(q,t; s,0)\dot{u}_i(q,0)]d\Omega + \rho \int_{\Omega} [p_{ki}^*(q,t; s,0)\dot{u}_i(q,0)]d\Omega
\]

where \( u_{ki}^* \) and \( p_{ki}^* \) are fundamental values and \( C_{ki} \) is an independent free term, similar to that used for elastostatics formulations.

It is important to mention that, in eqn (3), only the fundamental values were considered of quiescent past and that the principle of time translation should be applied for external collocations as shown by Coda & Venturini \([12]\).

### 3 Fundamental solutions

The Stokes general solution expression is given by:

\[
u_{ki}^*(q,\tau; s/f) = \frac{1}{4\pi\rho} \left\{ \frac{3r_rk}{r^3} - \delta_{ki} \right\} \int_{\Gamma} \alpha f(\tau - \alpha r) d\alpha + \\
+ \frac{r_rk}{r^3} \left[ \frac{1}{C_1^2} f(\tau - \frac{r}{C_1}) - \frac{1}{C_2^2} f(\tau - \frac{r}{C_2}) \right] + \frac{\delta_{ki}}{rC_2} f(\tau - \frac{r}{C_2})
\]
\[ p^*_k(q, \tau, s / f) = \frac{n_j}{4\pi} \left\{ -6C_s^2 \left[ \frac{r_i r_j r_k}{r^5} - \frac{\delta_{ij} r_k + \delta_{ik} r_j + \delta_{jk} r_i}{r^3} \right] \int_{C_i^3}^{C_i^1} \alpha f (\tau - \alpha r) d\alpha + \right. \]

\[ + 2 \left[ \frac{r_i r_j r_k}{r^5} - \frac{\delta_{ij} r_k + \delta_{ik} r_j + \delta_{jk} r_i}{r^3} \right] \int f \left( \tau - \frac{r}{C_2} \right) - \left( \frac{C_2^2}{C_1^2} \right) f \left( \tau - \frac{r}{C_1} \right) \right] + \]

\[ + 2 \left[ \frac{r_i r_j r_k}{r^4 C_2} \right] \int \left[ f \left( \tau - \frac{r}{C_2} \right) - \frac{C_2}{C_1} f \left( \tau - \frac{r}{C_1} \right) \right] \left[ 1 - 2 \left( \frac{C_2}{C_1} \right) f \left( \tau - \frac{r}{C_2} \right) + \frac{r}{C_1} f \left( \tau - \frac{r}{C_1} \right) \right] \]

\[ - \frac{\delta_{ik} r_j + \delta_{jk} r_i}{r^3} \left[ f \left( \tau - \frac{r}{C_2} \right) + \frac{r}{C_2} f \left( \tau - \frac{r}{C_2} \right) \right] \right] \right\} \]  \hspace{1cm} (4b)

where the derivative with respect to time is denoted by "\( \dot{f} \)".

It is easy to note that the time behaviour of this expression depends upon the adopted \( f(\tau) \).

In order to derive the first 3D time domain BEM formulation, Karabalis and Beskos [1] have adopted:

\[ b^*_k = \delta(\tau) \delta(q - s) \delta_{ki} \]  \hspace{1cm} (5)

The resulting fundamental values, obtained by using the load function of eqn (5), exhibit terms in "\( \delta(\ )\)" and "\( \dot{\delta}(\ )\)", what make it very difficult to treat initial conditions and to deal with integral displacement representation using constant time approximation. For the stress integral representation, terms in "\( \ddot{\delta}(\ )\)" are present, consequently increasing the difficulties of dealing with this approach. It should be emphasised that the presence of "\( \ddot{\delta}(\ )\)" and "\( \dot{\delta}(\ )\)" at the fundamental values requires linear and quadratic approximations in time, what enforces the use of four and six new matrices, respectively, for each time step, to follow conveniently the time marching process.

An alternative approach, due to Coda [6], is based on assuming the Heaviside time behaviour for the load function, i.e.:

\[ b^*_k = [H(\tau) - H(\tau - \Delta t)] \delta(q - s) \delta_{ki} / \Delta t \]  \hspace{1cm} (6)

In this case, the displacement representation can be treated by assuming constant time approximation, with a very good convergence range [5]. As it can be seen, for the displacement fundamental value only Heaviside terms are present, therefore the initial velocity expression can be integrated without difficulties. Due to the presence of Dirac's delta in the traction fundamental value, initial displacement conditions and the stress integral representation can not be conveniently implemented when using constant time approximation.
Taking into account the above comments, it is obvious that a smoother fundamental solution gives more generality to the time domain BEM formulation. In this sense, the load time behaviour, to be adopted here, arises from the follow expression:

\[
g(\tau) = \left[ 56\left(\frac{\tau}{R_t} - \frac{1}{2}\right)^7 - 36\left(\frac{\tau}{R_t} - \frac{1}{2}\right)^5 + 11\left(\frac{\tau}{R_t} - \frac{1}{2}\right)^3 + \frac{\tau}{R_t} \right] H(\tau) - H(\tau - R_t) + H(\tau - R_t) \tag{7}
\]

where "\(\tau\)" is the time and "\(R_t\)" is a portion of "\(\Delta t\)" (time interval) given by:

\[
R_t = \Delta t / RN
\]

with "RN" being any positive real number.

In figure 1, the curve representing the function of eqn (7), obtained when "RN \(\geq 2\)", is shown to illustrate its behaviour.

\[
Figure 1: "g(\tau)" function for "RN \(\geq 2\)"
\]

In order to achieve a finite number of matrices, it is desirable that all fundamental values, associated to a specific load function, vanish after some period of time. Thus, an appropriate load function \(f(\tau)\), exhibiting this behaviour, can be found out by the difference between two \(g(\tau)\) functions defined at different phases, as follows,

\[
f(\tau) = \frac{g(\tau) - g(\tau - Rd * R_t)}{Rd} \tag{9}
\]

where "\(Rd \geq 1\)" is a parameter used to define the temporal unit impulse length.

For the cases taken when "\(Rd=1\)" and "\(Rd=RN-1\)"; the load function behaviours are illustrated in figures (2) and (3) respectively.

\[
Figure 2: Values of "f(\tau)" for "Rd = 1"
\]
It is clear that when \( \text{R}_t \to 0 \), expression (9) turns into eqn (5) for \( \text{RD} = 1 \), and into eqn (6) for \( \text{Rd} = \text{RN} - 1 \). It is important to mention that \( \text{Rd} \) can be greater than \( \text{RN} - 1 \); in this situation, the unit impulse is distributed over more than one time step.

The final fundamental solution expression for the proposed load function can be represented directly by expression (4). For this purpose, the following terms are the only ones required in their explicit form,

\[
\int_{c^1_t} \int_{c^1_r} a(r - \alpha) d\alpha = \frac{A_k}{r^2(r+1)(k+2)} \left[ \left( \frac{k+2}{2} \right) \left( \frac{\tau}{\text{Rt}} - \frac{1}{2} \alpha \frac{\tau}{\text{Rt}} \right) \left( \frac{\tau}{\text{Rt}} - \frac{(k+1)\alpha}{\text{Rt}} \right) \right] +
- \frac{\Delta t}{2^{k+1}} \left[ \frac{1}{2} H(\tau - \alpha) - \left( \frac{k+3}{2} \right) H(\tau - \alpha - \text{Rt}) \right] + \frac{1}{6r^2} \left[ \frac{3}{\text{Rt}} (\alpha^2 r^2 - (\tau - \text{Rt})^2) \right] +
- (3\tau - 2\text{Rt}) H(\tau - \alpha - \text{Rt}) - \frac{(\tau - \alpha)^2}{\text{Rt}} \frac{(\tau - \alpha)}{\text{Rt}} \left[ H(\tau - \alpha) - H(\tau - \alpha - \text{Rt}) \right]
\]

(10)

and

\[
\ddot{g}(\tau - \alpha) = \left[ \frac{392}{\text{Rt}} \left( \frac{\tau - \alpha}{\text{Rt}} - \frac{1}{2} \right)^6 - \frac{180}{\text{Rt}} \left( \frac{\tau - \alpha}{\text{Rt}} - \frac{1}{2} \right)^4 + \frac{33}{2 \text{Rt}} \left( \frac{\tau - \alpha}{\text{Rt}} - \frac{1}{2} \right)^3 + \frac{1}{\text{Rt}} \right]
\left[ H(\tau - \alpha) - H(\tau - \alpha - \text{Rt}) \right]
\]

(11)

The right hand side of expression (10) represents the summation for \( k=3,5,7 \), while the constants \( A_k \) assume the following values:

\[ A_3 = \frac{11}{2} ; A_5 = -36 ; A_7 = 56 \]

(12)

4 Numerical aspects:

When assuming the “Smooth” fundamental solution, in equation (3), for \( 1 \leq \text{Rd} \leq \text{RN} - 1 \) one achieves,
The equations are given by:

\[ C_{ki}(Q,s) \int_{t-\Delta t}^{t} \frac{u_{i}(s,\tau)}{\Delta t} d \tau = \int_{\Gamma} \int_{0}^{t} u_{i}^{*}(Q,t; s, \tau) p_{i}(Q, \tau) d \Gamma d \tau + \]

\[ - \int_{0}^{t} \int_{\Omega} u_{i}(Q, \tau) p_{i}^{*}(Q,t; s, \tau) d \Omega d \tau + \int_{0}^{t} \int_{\Omega} u_{i}^{*}(Q,t; s, \tau) b_{i}(Q, \tau) d \Omega d \tau + \]

\[ + \rho \int_{\Omega} \left[ u_{i}^{*}(Q,t; s,0) \hat{u}_{i}(Q,0) \right] d \Omega + \rho \int_{\Omega} \left[ u_{i}^{*}(Q,t; s,0) \hat{u}_{i}(Q,0) \right] d \Omega \]

(13)

As it is usual for the time domain BEM formulation, expression (13) can be numerically treated by adopting appropriate space and time discretizations. For this fundamental solution no restriction is made on the choice of these discretizations. Using constant time approximation for both displacements and tractions, and adopting, for spatial discretization, isoparametric quadratic boundary elements and isoparametric quadratic cells, expression (13) becomes:

\[ C_{ki}(s) U_{i}(s, t)^{N_t} = \int_{\Gamma(j)} \int_{t_{\theta-1}}^{t_{\theta}} u_{i}^{*}(Q,t; s, \tau) \phi_{a}^{j}(Q) \psi_{a}^{\theta} d \tau d \Gamma P_{i,j}^{a} + \]

\[ - \int_{\Gamma(j)} \int_{t_{\theta-1}}^{t_{\theta}} p_{i}^{*}(Q,t; s, \tau) \phi_{a}^{j}(Q) \psi_{a}^{\theta} d \tau d \Gamma U_{i,j}^{a} + \int_{\Omega(t,j_{\theta-1})}^{t_{\theta}} u_{i}^{*}(Q,t; s, \tau) \phi_{a}^{\ell}(Q) \psi_{a}^{\ell} d \tau d \Omega b_{i,j}^{a} \]

\[ + \rho \int_{\Omega(t)} \hat{u}_{i}^{*}(Q,t; s,0) \phi_{a}^{\ell}(Q) d \Omega V_{i,j}^{a} + \rho \int_{\Omega(t)} \hat{u}_{i}^{*}(Q,t; s,0) \phi_{a}^{\ell}(Q) d \Omega U_{i,j}^{a} \]

(14)

where \( \theta \) varies from 1 to the time step number \( N_t \), and the density approximations are given by,

\[ U_{i,\theta}(Q,j,t) = \phi_{a}^{(j)} \psi_{a}^{(\theta)} U_{i,j}^{a} ; \quad P_{i,\theta}(Q,j,t) = \phi_{a}^{(j)} \psi_{a}^{(\theta)} P_{i,j}^{a} \]

\[ B_{i,\theta}(Q',t) = \phi_{a}^{(j)} \psi_{a}^{(\theta)} b_{i,j}^{a} ; \quad V_{i,\theta}(Q',t) = \phi_{a}^{(j)} V_{i,j}^{a} ; \quad V_{i,\theta}(Q',t) = \phi_{a}^{(j)} V_{i,j}^{a} \]

(15)

for which the superscripts \( j \) and \( \ell \) indicate the adopted boundary element and internal cell, respectively, while \( \phi_{a}^{(j)} \) and \( \phi_{a}^{(\ell)} \) are the spatial shape functions and \( \psi_{a}^{(\theta)} (\psi = 1) \) is the time approximation.

After performing all time convolutions and spatial integrals indicated in eqn (14), one achieves:

\[ H^{\theta} U^{\theta} = G^{\theta} P^{\theta} + B^{\theta} b^{\theta} + W^{N} V^{0} + S^{N} U^{0} \]

(16)

where \( \theta = 1, 2, 3,..., N_t \).

Eqn (16) is the usual time domain BEM algebraic system of equations. It is worth to mention that the spatial integration has been performed by adopting an element subdivision proportional to \( Rd * C_{1} \), to guarantee an accurate
evaluation of the kernel. This subdivision should be enough fine near the load point and coarser far from this point.

5 Example

In the example selected to illustrate the presented formulation, a prismatic rod is analysed when subjected to an impact at the free end. Figure 4 gives the geometry and the material properties adopted for this case, as well as the loading characteristics. Thirty four quadratic elements were taken to discretize the solid boundary, as shown in Figure 5. It has been adopted outside collocations distant “C_1\Delta t / 15” from the boundary and the fundamental load function given by assuming Rn = 15 and Rd = 14. For \Delta t = 0.006s, a 5x5 sub-element division was the finest one employed. The same spatial integral scheme has been used for both “Smooth” and “Heaviside” fundamental solutions.

\[
\begin{align*}
E &= 11000 \text{ kg/cm}^2 = 11 \text{ kgf/cm}^2 \\
\rho &= 0.002 \text{ kg/cm}^3
\end{align*}
\]

Figure 4: Geometry, material properties and load behaviour.

Figure 5: Discretization of the rod.

The free end displacement and the constrained end reaction values are compared in figures 6 and 7, respectively. As it can be seen, the results of both formulations agree very well with the analytical solution and present the same behaviour with respect to the numerical dumping of this kind of formulation. It has been verified, but not exhibited here to save space, that the “Smooth” approach presents the same convergence range as the “Heaviside” one [5].
6 Conclusions

A powerful fundamental solution to be adopted for 3D time domain BEM formulations has been presented and discussed. Due to the good results exhibited and the smoothness of this fundamental solution, it is obvious that the stress integral representation and the initial values expressions can be successfully implemented. Therefore, this "Smooth" fundamental solution represents a great step to establish definitively the time domain BEM formulation, making this approach a reliable tool for transient elastodynamic analysis.


