Smooth modelling of geometry in BEMs
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Abstract

In this paper, we present the discussion of the shortcomings of modelling the boundary geometry by the Lagrange-type elements as well as by the standard Overhauser elements. The modified Overhauser elements are developed with removing the shortcomings associated with standard elements and preserving the C^1 continuity. Several illustrative examples are considered within a numerical experiment.

1 Introduction

Recently a great progress has been achieved in the development of advanced BEM formulations especially with respect to regularized formulations facilitating an accurate numerical evaluation of singular and nearly singular integrals^1,2. In the regularization, there have been revealed strong requirements on continuity of the approximated boundary densities^3 and boundary geometry^4,5. It should be stressed that the continuity requirements are accepted more widely for approximation of boundary densities than for boundary geometry. The latter is modelled very often by approximating the global coordinates of boundary points using the standard Lagrange elements (with interpolation by Lagrange polynomials). It is well known that such approximations give rise to some continuity troubles. Although the approximated coordinates are continuous everywhere, their derivatives (such as the tangent vector and curvature) are discontinuous at the element junctions, in general. The origin of such discontinuities consists in an insufficiently accurate approximation of the primary quantity (coordinates). Thus, in addition to the nonunique definition of the derivative quantities (tangent vector, curvature) an undesirable inaccuracy appears in such approximations. Usually, this difficulty in
2-d problems would like to be resolved by using \( C^1 \)-continuous Overhauser elements\(^6\). In spite of the continuity of tangent vectors approximated by standard Overhauser (SO\( \nu \)) elements, the accuracy of such an approximation fails rapidly if the contour arcs over two neighbouring elements are not symmetric with respect to the normal vector at the element junction. Thus, the nonequidistant partitioning of the boundary contour or an abrupt change of the curvature yield failure of the approximation by SO\( \nu \) elements\(^7\).

Making use of the interpolation polynomials employed in the SO\( \nu \) elements with improving the determination of the tangent vector at the element junction, we derive in this paper the modified Overhauser (MO\( \nu \)) elements which are \( C^1 \)-continuous and got off the shortcoming associated with the SO\( \nu \) elements. Combination of various curves over two neighbouring elements are considered in a numerical example.

**2 Definition of geometrical characteristics**

Consider two segments \( \Gamma_A \) and \( \Gamma_B \) forming a portion of the boundary contour \( \Gamma \). In general, the point \( \zeta = \Gamma_A \cap \Gamma_B \) is admitted to be a corner point. Otherwise, the segments \( \Gamma_A \) and \( \Gamma_B \) are assumed to be smooth curves with unique definition of the tangent and normal vector. Choosing the coordinate system as shown in Fig.1, one can define the points on \( \Gamma_A \) and \( \Gamma_B \), respectively, as

\[ \Gamma_A: y = g(x), \; x \in (-\alpha, 0), \quad \Gamma_B: y = f(x), \; x \in (0, -2B\alpha) \]

where both \( g \) and \( f \) are differentiable functions at least up to the second order on the intervals \( x \in (-2\alpha, 0) \) and \( x \in (0, 2B\alpha) \), respectively.

The differential characteristics on \( \Gamma_A \cup \Gamma_B \), such as the tangent vector and curvature are given by the first and second order derivatives, respectively. Discontinuities of both these quantities are admissible at \( x = 0 \). In spite of this discontinuity of the tangent vector, the difference of the tangent vectors can play the role of a regularizing factor in BEM formulations\(^1\). A typical regularized integral of strongly singular kernel is given as
If the tangent vector is continuous at the element junction, then \( t_i^A (\zeta) = t_i^B (\zeta) = t_i (\zeta) \). Recall that an accurate computation of the integral given by eq. (1) requires sufficiently accurate approximations not only for the tangent vector along \( \Gamma_A \) and \( \Gamma_B \), but also for curvature in the near vicinity of the singular point \( \zeta \), since

\[
\lim_{x \to 0} \frac{t_i (x) - t_i^P (0)}{x} = \left. \frac{dt_i}{dx} \right|_{\Gamma_p} (0)
\]

Hence, it should be stressed that the advanced BEM formulations require a progressive modelling of the boundary geometry obeying a sufficient accuracy for coordinates of the boundary points and their derivatives up to the second order. Furthermore, in a consistent modelling of geometry, the tangent vectors and curvature are derived from the approximated coordinates. Concluding, the coordinates of the boundary points in advanced BEM formulations should be approximated by polynomials of the second order at least.

### 3 Modelling by standard elements

The most widely used elements for modelling of the boundary geometry are the Lagrange-type elements with using the Lagrange polynomials for interpolation within the approximation element.

#### Lagrange-type elements

Such an element is defined by \( n+1 \) nodal points (two of them are located at the element ends) and the Lagrange polynomials of the \( n \)-th order \( N^a (\xi) \), \( (a = 1, 2, \ldots, n+1) \). In the consistent geometry modelling, the approximated coordinates, tangent vector and curvature are given as

\[
\eta_i \big|_{\Gamma_p} = \sum_{a=1}^{n+1} \eta_i^{ap} N^a (\xi)
\]

\[
t_i (\xi) \big|_{\Gamma_p} = h_i^P (\xi) / h_i^P (\xi), \quad h_i^P (\xi) = \sum_{a=1}^{n+1} \eta_i^{ap} N^a (\xi), \quad h^P = \sqrt{h_i^P h_i^P}
\]

\[
k(\xi) \big|_{\Gamma_p} = (1 / h_i^P (\xi))^3 \epsilon_{ijk} h_j^P (\xi) a_k^P (\xi), \quad a_k^P (\xi) = \sum_{a=1}^{n+1} \eta_i^{ap} N^a (\xi)
\]

where \( \eta_i^{ap} \) are the global coordinates of the nodal points.

The internal nodal points are not required to be distributed equidistantly. In the case of isoparametric elements \( \xi \in (-1, 1) \) and the nodal points are distributed equidistantly. These conforming elements obey \( C^\infty \) continuity in the interior points but only \( C^0 \) continuity across the element boundary, in general.

Due to the continuity of the approximated tangent vector and curvature in the
interior of the element it is unreasonable to include the points of actual
discontinuities of these quantities into the interior of Lagrange elements.
Although it is easy to identify the corner points (discontinuities of the tangent
vector), the discontinuities of the curvature are not visible, in general. Note that
(n+1) polynomials $N^p$ can fit exactly the curved line described over the element
by a polynomial of the m-th order with $m \leq n$. Such a case is not met, if the
smooth curve over the element consists of a finite number of arcs described by
polynomials of the orders $m_k \leq n$. Likewise, if the curve over the element is
given by a polynomial (with constant coefficients) of the order $m > n$, the fitting
is not exact any more and the discontinuities of the approximated tangent vector
can arise at the element junctions. Thus, the Lagrange elements are applicable to
the modelling of corners.

Consider a smooth curve approximated by the standard Lagrange element,
the length of which is shrunk to zero. The element becomes more straight the
shorter the length of the element. For simplicity, we can consider an
isoparametric element with the distance between any two nodes being denoted
as $B$ and approaching $B$ to zero. Then, selecting the isoparametric axis parallel
with $t_i^e(\eta_i^1p)$, we may write

$$\eta_i^{ap} = \eta_i^{1p} + aBt_i^e(\eta_i^1p) + \xi(\eta_i^{1p})O(B^2)$$

Hence, and from (6) with using $\sum a N^a(\xi) = 0$, one obtains

$$\lim_{B \to 0} t_i(\eta^{1p})|_{\Gamma_p} = \lim_{B \to 0} \frac{h_i^p(-1)}{h_i^p(-1)} = t_i^e(\eta_i^{1p})$$

which is not a surprising result.

Finally, assuming the curve shown in Fig. 1 to be smooth, we can approximate
the segments $\Gamma_A$ and $\Gamma_B$ by the quadratic Lagrange element $\Gamma_p$ determined with
the nodal points $\eta_1^{1p} = \eta_2^{2A}$, $\eta_2^{2p} = \eta_3^{2A}$, and $\eta_3^{3p} = \eta_3^{3B}$ Taking the
parametric axis identical with the x-axis, we have

$$\eta_1|_{\Gamma_p} = \alpha \xi, \quad \eta_2|_{\Gamma_p} = \beta \xi, \quad \eta_3|_{\Gamma_p} = \sum a \eta_2^{ap} N^a(\xi), \quad \xi \in (-1, B)$$

$$h_1^p(\xi) = \alpha, \quad h_2^p(\xi) = \beta = 0 = \frac{B}{1 + B} \eta_2^{1p} - \frac{1 - B}{B} \eta_2^{2p} + \frac{1}{B(1 + B)} \eta_2^{3p}$$

Hence,

$$\lim_{B \to 0} h_i^p(\xi = 0) = \alpha \delta_{i1} + \beta \delta_{i2} \lim_{B \to 0} \frac{\eta_2^{3p} - \eta_2^{2p}(1 - B^2)}{B(1 + B)}$$

$$= \alpha \delta_{i1} + \beta \delta_{i2} f'(0+) - t_i^e(\eta_i^{2p})$$

Thus,

$$\lim_{B \to 0} t_i^p(\eta_i^{3A}) = t_i^e(\eta_i^{3A})$$
The same result can be derived if we employ the quartic Lagrange element $\Gamma_p$ determined with the nodal points $\eta^{1p} = \eta^{1A}$, $\eta^{2p} = \eta^{2A}$, $\eta^{3p} = \eta^{3A}$, $\eta^{4p} = \eta^{3B}$, $\eta^{5p} = \eta^{4B}$ for approximation of the smooth curve shown in Fig. 1.

**Standard Overhauser elements**

These elements are developed on a parametric cubic interpolation by parametric quadratics $^6$. The approximation is confined to the region between the second and third nodes, while information is taken also from the outer nodes. That is why $C^1$ continuity is achieved at element junctions, but also why standard Overhauser elements are applicable only to approximation of smooth portions of the boundary contour. The expressions for the approximated global coordinates, tangent vector and curvature can be obtained from eqs. (3)-(5) by taking $n = 4$ and replacing the polynomials $N^a(\xi)$ by $M^a(\xi)$ given as

\[
\begin{align*}
M^1 &= -\xi (\xi - 1)^2 / 2 \\
M^2 &= (\xi - 1)(3\xi^2 - 2\xi - 2) / 2 \\
M^3 &= -\xi (3\xi^2 - 4\xi - 1) / 2 \\
M^4 &= \xi^2 (\xi - 1) / 2
\end{align*}
\]

with $\xi \in(0,1)$.

In order to illustrate the notation of nodal points, we may employ Fig.1 provided that the curve given by functions $g(x)$ and $f(x)$ for $x \in(-2\alpha, 2\alpha)$ is smooth. Note that $\eta^{4A} = \eta^{3B}$ for $\Gamma_A$ and $\eta^{1B} = \eta^{2A}$, $\eta^{2B} = \eta^{3A}$ for $\Gamma_B$.

Although the interpolation polynomials are cubic ones, the Overhauser elements can fit exactly only an arc described by a polynomial of the $m$-th order with $m \leq 2$. In addition to inaccuracies arising due to approximation of higher than quadratic polynomials, the nonequidistant distribution of nodal points can also give rise to deviations of the curves fitted by Overhauser elements from the exact curves.

In order to distinguish the notations for the Lagrange and Overhauser elements, we replace $h^i_p$ by $H^i_p$ in the case of the Overhauser elements. According to the definition of the standard Overhauser element and the notations shown in Fig. 1, we may write

\[
H^i_A(\xi = 1) = (\eta^{4A}_i - \eta^{2A}_i) / 2 = (\eta^{3B}_i - \eta^{1B}_i) / 2 = H^i_B(\xi = 0) = (\eta^{3B}_i - \eta^{2A}_i) / 2 = H^{SOv}_i
\]

and further,

\[
H^{SOv}_i = \frac{B+1}{2} \alpha_1 \delta_{si} + \frac{1}{2} \left[ f(B\alpha) - g(-\alpha) \right] \delta_{s2}, \quad i^{SOv}_i(\eta^{3A}) = H^{SOv}_i / H^{SOv}
\]

Thus, in general, the value of $i^{SOv}_i(\eta^{3A})$ is dependent on the choice of $B$ (position of the node $\eta^{3B}$). Similar dependence can be observed for any quantity approximated by standard Overhauser elements.

Depending on the functions $g(x)$ and $f(x)$, one can find none, one or more values of $B$ satisfying the equation

\[
i^{SOv}_i(\eta^{3A}) = i^{ex}_i(\eta^{3A}), \quad \text{where} \quad i^{ex}_i(\eta^{3A}) = \frac{\delta_{s1} + f'(0+)\delta_{s2}}{1 + (f'(0+))^2}
\]
with $f'(0+) = g'(0-)$. Note that eq. (9) cannot be satisfied for arbitrary $B$, unless $f(x) = g(x) = kx$ (straight segments). In contrast to the Lagrange elements

$$\lim_{B \to 0} t_i^{SO\nu}(\eta^{3A}) = \frac{\delta_{i1} - g(-\alpha) / \alpha \delta_{i2}}{\sqrt{1 + (g(-\alpha) / \alpha)^2}} \neq t_i^{ex}(\eta^{3A})$$

unless $g(x) = kx$ with $g'(0-) = f'(0+)$. 

### 4 Modified Overhauser elements 

The modified Overhauser elements have been proposed and employed for approximation of boundary densities over two segments involving a corner with obeying the continuity of gradients of the approximated field. Now, we propose modified Overhauser elements for modelling the boundary geometry with obeying the continuity of the tangent vector on smooth boundary contour including the element junctions. The need for the development of such elements is evident. Although the standard Lagrange elements can be used for modelling a corner, they do not guarantee the continuity of the tangent vector at their other ends if required. On the other hand, the standard Overhauser elements are inappropriate for modelling a corner as well as for approximation over two smoothly curved segments with different lengths of elements.

The idea of construction of the modified Overhauser elements consists of the primary determination of the tangent vector at the element junction as accurately as possible and the secondary definition of fictitious nodal points behind the joint point. In the case of two elements $\Gamma_A$ and $\Gamma_B$ shown in Fig. 1, the fictitious nodal points $\eta^{4A}$ and $\eta^{1B}$ are not visualized. Now, we shall consider separately two cases, when the joint point $\zeta = \Gamma_A \cap \Gamma_B$ is a corner or is not.

**Corner at $\zeta$**

The tangent vector $t_i^A(\zeta)$ can be determined by using the standard Lagrange element $(1A, 2A, 3A)$. The cartesian coordinates of the fictitious node $4A$ are given by

$$\eta_i^{4A} = \eta_i^{2A} + 2H_i^A(1), \quad H_i^A(1) = t_i^A(\zeta)A^p, \quad A^p = t_j^A(\zeta)(\eta_j^{3A} - \eta_j^{2A})$$

Thus, the modified Overhauser element $\Gamma_A$ is defined by the nodes $(1A, 2A, 3A, 4A)$ and the interpolation polynomials $M^A(\zeta)$ given by eq. (8) for $\zeta \in <0,1>$. Similar procedure can be employed also in the definition of the modified Overhauser element $\Gamma_B$ specified by the nodes $(1B, 2B = 3A, 3B, 4B)$ with the cartesian coordinates of the fictitious node $1B$ being given by

$$\eta_i^{1B} = \eta_i^{3B} - 2H_i^B(0), \quad H_i^B(0) = t_i^B(\zeta)B^p, \quad B^p = t_j^B(\zeta)(\eta_j^{3B} - \eta_j^{3A})$$

where the tangent vector $t_i^B(\zeta)$ is determined by using the standard Lagrange element $(3A, 3B, 4B)$.

Note that if the left neighbour element to $\Gamma_A$ is the standard Overhauser element, then the tangent vector at $\eta^{2A}$ can be obtained also by using the
standard Lagrange element \((1A, 2A, 3A)\) for approximation over \(\Gamma_A\) with restricting \(\xi\) to the interval \(<0, 1>\). But if the lengths of these two neighbouring elements are different, it is inappropriate to employ the standard Overhauser element for approximation over neighbouring element and one should consider the connection of two modified Overhauser elements on a smooth contour arc.

**Smooth contour arc at \(\zeta\)**

In the case of a general smooth contour, the determination of the tangent vector \(t_i(\zeta)\) by standard Lagrange elements might be inaccurate. But there are several possibilities how to determine \(t_i(\zeta)\) more accurately.

**Lav** : In this approach, we determine the direction of \(t_i(\zeta)\) as the average of two directions given by \(t_i^A(\zeta)\) and \(t_i^B(\zeta)\) defined by the standard Lagrange elements \((1A, 2A, 3A)\) and \((3A, 3B, 4B)\), respectively. Thus,

\[
t_i^{Lav}(\zeta) = \frac{\tau_i}{\sqrt{\tau_1^2 + \tau_2^2}}, \quad \tau_i = t_i^A(\zeta) + t_i^B(\zeta)
\]

Note that such a determination of \(t_i(\zeta)\) is consistent with the assumptions used in the BEM formulations.

**L2in** : Now, the tangent vector is determined at the interior nodal point of the standard quadratic Lagrange element \((2A, 3A, 3B)\) as

\[
t_i^{L2in}(\zeta) = \frac{h_i^{L2}(0)}{h_i^{L2}(0)}, \quad h_i^{L2}(0) = -\frac{B}{1+B} \eta_i^{2A} - \frac{1-B}{B} \eta_i^{3A} + \frac{1}{B(1+B)} \eta_i^{3B}
\]

**L4in** : In comparison with L2in approach, the only difference consists in the use of the standard quartic Lagrange element \((1A, 2A, 3A, 3B, 4B)\). Then,

\[
t_i^{L4in}(\zeta) = \frac{h_i^{L4}(0)}{h_i^{L4}(0)}, \quad h_i^{L4}(0) \sim \frac{B^3}{2} (2B+1) \eta_i^{1A} - 4B^3(B+2) \eta_i^{2A}
\]

\[
+ \frac{3}{2} (B^2 - 1)(B+2)(2B+1) \eta_i^{3A} + 4(2B+1) \eta_i^{3B} - \frac{B+2}{2} \eta_i^{4B}
\]

Recall that both the \(t_i^{L2}(\zeta)\) and \(t_i^{L4}(\zeta)\) converge to \(t_i^{ex}(\zeta)\) as \(B \to 0\). Furthermore, the use of L2in and/or L4in approach yields continuous curvature at \(\zeta\). On the other hand, \(t_i^{Lav}(\zeta)\) does not necessarily converge to \(t_i^{ex}(\zeta)\) as \(B \to 0\), though \(t_i^B(\zeta)\) does, and the curvature is usually discontinuous at \(\zeta\).

Concluding, L2in and L4in approaches seems to be more convenient for approximation of curves with continuous curvature at \(\zeta\), while the Lav approach is better for smooth curves with discontinuous curvature at \(\zeta\).

Finally, the cartesian coordinates of the fictitious nodes are given as

\[
\eta_i^{4A} = \eta_i^{2A} + 2H_i^{A}(\xi = 1), \quad \eta_i^{1B} = \eta_i^{3B} - 2H_i^{B}(\xi = 0)
\]

where

\[
H_i^{A}(\xi = 1) = t_i^{MOV}(\zeta) A^P, \quad A^P = t_j^{MOV}(\zeta) (\eta_j^{3A} - \eta_j^{2A})
\]

\[
H_i^{B}(\xi = 1) = t_i^{MOV}(\zeta) B^P, \quad B^P = t_j^{MOV}(\zeta) (\eta_j^{3B} - \eta_j^{3A})
\]
with $t_{i}^{MOv}(\zeta)$ being the tangent vector controlling the accuracy of the approximation by the modified Overhauser elements. In this paper, we have proposed three different selections of $t_{i}^{MOv}(\zeta)$ according to $MOv \in \{Lav, L2in, L4in\}$.

5 Numerical examples

In order to illustrate various approximation techniques employed in modelling the geometry of smooth curves, we have considered a smooth junction of two arcs defined by the polynomials of various orders. For conciseness, we present only the numerical results for the angular deviations, $\delta$, of the approximate tangent vector $t_{i}(\zeta)$ from the exact one versus $B$, which is the ratio of the lengths of two arcs $\Gamma_{A}$ and $\Gamma_{B}$. We have used five different approaches for modelling the geometry. These are: SLag - standard Lagrange elements over $\Gamma_{A}$ and $\Gamma_{B}$ separately, SOv - standard Overhauser elements, MOvLa - modified Overhauser elements with $t_{i}^{MOv}(\zeta) = t_{i}^{Lav}(\zeta)$, MOvL2in - MOv elements with $t_{i}^{MOv}(\zeta) = t_{i}^{L2}(\zeta)$, and MOvL4in - MOv with $t_{i}^{MOv}(\zeta) = t_{i}^{L4}(\zeta)$. The presented numerical results correspond to the following combinations of the joined curves:

(a) $\Gamma_{A}$ - straight line $y = 0$, $\Gamma_{B}$ - circular arc $y = r - \sqrt{r^2 - x^2}$. Then,
\[ t_{i}^{ex}(\zeta) = \delta_{i1}, \]
\[ \kappa^{ex}(\zeta-) = 0 \neq \kappa^{ex}(\zeta+) = 1/r \]
(b) $\Gamma_{A}$ - quadratic parabola $y = gx^2$, $\Gamma_{B}$ - cubic parabola $y = kx^3$. Then,
\[ t_{i}^{ex}(\zeta) = \delta_{i1}, \]
\[ \kappa^{ex}(\zeta-) = 2g \neq \kappa^{ex}(\zeta+) = 0 \]
(c) $\Gamma_{A}$, $\Gamma_{B}$ - circular arcs $y = r - \sqrt{r^2 - x^2}$. Then,
\[ t_{i}^{ex}(\zeta) = \delta_{i1}, \quad \kappa^{ex}(\zeta) = 1/r \]
(d) $\Gamma_{A}$, $\Gamma_{B}$ - cubic parabolas $y = kx^3$,
\[ t_{i}^{ex}(\zeta) = \delta_{i1}, \quad \kappa^{ex}(\zeta) = 0 \]
(e) $\Gamma_{A}$ - quartic parabola $y = gx^4$, $\Gamma_{B}$ - cubic parabola $y = kx^3$. Then,
\[ t_{i}^{ex}(\zeta) = \delta_{i1}, \quad \kappa^{ex}(\zeta) = 0 \]
Fig. 2 (b)

Fig. 2 (c)

Fig. 2 (d)

Fig. 2 (e)
Conclusions

The modified Overhauser elements have been developed for better modelling the boundary geometry in 2-d problems. There is considered a smooth connection of two segments as well as connection at a corner. The modelling is consistent because the approximations for the tangent vector and curvature are obtained from that of the global coordinates of the points on the contour by differentiations. The accuracy of all these approximations are controlled by the accuracy of the determination of the tangent vector at the junction of two elements. It has been revealed from the numerical analysis that the averaging approach (MOvLav) is the best if the curvature is discontinuous at the joint point, while the MOvL2in and MOvL4in are better if the curvature is continuous at the joint point.

References