Two-dimensional linearly-layered potential flow by boundary elements

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Abstract

A verification of a recently developed boundary element formulation for linearly varying material media in Poisson type problems is presented. Examples illustrate the accuracy and simplicity of the new approach. Also, some novel modelling ideas concerning piecewise continuous and impermeable media are advanced.

1 Introduction

Recently, Shaw\textsuperscript{1} presented a new formulation for finding Green's functions specifically designed to analyze potential problems occurring in heterogeneous media. Convenient boundary integral representations of such problems constitute important theoretical and practical areas in boundary element evolution because they have traditionally been perceived as a weakness of boundary element methods (BEM) when compared to domain-based numerical methods. This is because, until just recently, there has seemed to be no way to avoid an explicit domain discretization when material properties vary continuously with position in space.

This paper presents the BEM formulation and worked examples for the special case of a two-dimensional linearly-varying material governed by a Poisson-type equation.

2 Formulation

The example problems presented in this paper may be interpreted, among other things, as steady-state heat-flow or groundwater flows. We shall use these physical terminologies throughout the remainder of the paper.
Consider a two-dimensional \((x,y)\) material which has a varying conductivity in the \(x\)-direction, \(k(x)\). In the presence of internal heat generation \(Q(x,y)\), application of Fourier's Law leads to the governing equation for the temperature \(\phi(x,y)\):

\[
\frac{\partial}{\partial x} \left( k(x) \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(x) \frac{\partial \phi}{\partial y} \right) = -Q(x,y)
\]  

(1)

If the variation of conductivity is linear, then \(k(x) = k_1 x + k_0\), with \(k_0\) and \(k_1\) being constants \(k_1 \neq 0\) by hypothesis or else the problem reduces to the constant conductivity case. We may now invoke the simple coordinate transformations \(r = x + k_0/k_1\) and \(z = y\). Noting that \(\partial/\partial x = \partial/\partial r\) and \(\partial/\partial y = \partial/\partial z\), Equation (1) takes on the form

\[
\frac{\partial}{\partial r} \left( k_1 r \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial z} \left( k_1 r \frac{\partial \phi}{\partial z} \right) = -Q \left( r - \frac{k_0}{k_1}, z \right)
\]

which, upon division by \(k_1 r\), yields the suggestive form:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} = Q'(r,z).
\]  

(2)

where \(Q'(r,z) = -Q/k_1 r\). Equation (2) is the axisymmetric form of Poisson's equation in standard cylindrical coordinates \(r\) (radial) and \(z\). We may conclude therefore, that any two-dimensional potential problem derived from a linear relationship between flux and potential gradient (e.g., Fourier's Law, Darcy's Law, etc.), and with a unidirectional linear variation of conductivity may be recast as a standard Poisson's equation in axisymmetric cylindrical coordinates.

This result is, of course, applicable to any analytical or numerical technique available in the solution of linearly varying potential problems, though it would be expected that the domain-based finite difference and finite element methods would more naturally lend themselves to the direct approach of letting the material properties vary over each subregion of discretized space. This formulation is especially significant for the boundary element method, since ideally, domain discretization is always to be avoided if possible. Since the solution of Poisson's equation by BEM is a well-established technology\(^2\), the coordinate transformations \(r = x + k_0/k_1\) and \(z = y\) effectively transform a working axisymmetric BEM code into a tool for working the linearly-varying conduction problems.

The two-dimensional integral equation formulation is invariably cast over a planar region \(A\) bounded by a closed curve \(S\) as follows

\[
\phi(x,y) = \int_S \left( \phi^* \frac{\partial \phi}{\partial n} - \phi - \frac{\partial \phi^*}{\partial n} \right) dS + \iint_A Q' \phi^* dV
\]
where \( \partial / \partial n \) represents differentiation with respect to the outward pointing normal to the surface \( A \) and \( \varphi^* \) is the Green's function, which is in this case:

\[
\varphi^*(r, z, r_0, z_0) = \frac{K(\gamma)}{\pi \sqrt{(r + r_0)^2 + (z - z_0)^2}}
\]

for a source point at \((r_0, z_0)\). Here \( K(\gamma) \) is the complete elliptic integral of the first kind, with

\[
\gamma^2 = \frac{4rr_0}{\sqrt{(r + r_0)^2 + (z - z_0)^2}}
\]

For most commonly occurring forms of \( Q(x, y) \), the integral equations can be manipulated into a form devoid of domain integrals, and thus, the full power and convenience of BEM as a purely surface integral formulation may be exploited. Specifically, the very important special case with \( Q = 0 \) (Laplace's equation) lends itself to an exclusive surface integral relationship, as do all homogeneous elliptic partial differential equations.

We will now illustrate the validity of the method with pertinent examples.

3 Example 1

Figure 1a illustrates a simple one-dimensional problem which was cast in a two-dimensional geometry for the express design of testing the validity of the formulation with a tried and proven computer program that performs axisymmetric potential analysis. In nondimensional form, the geometry is a rectangle with a width equal to twice the height. Sides \( b-c \) and \( a-d \) are insulated, the potential on side \( a-b \) is specified as zero, while the potential on side \( c-d \) is equal to unity. The conductivity \( k \) varies with height and it is given by:

\[
k(y) = y + 0.1
\]

The exact solution to this problem is obtained as a trivial exercise in ordinary differential equations. Solving

\[
\frac{d}{dy} \left[ (y + 0.1) \frac{d \varphi}{dy} \right] = 0
\]

and applying the boundary conditions yields

\[
\varphi = \frac{\ln(10y + 1)}{\ln 11}
\]

To effect the boundary element solution, we first note that the coordinate transformation for this problem is given by:
This transformed problem that is actually used as data for the computer program is shown in Figure 1b. Note the new orientations of the sides in the transformed problem. A boundary element model was constructed with ten evenly-spaced linear elements per side. The results for both the exact analytical solution and the boundary element model are shown in the following table:

**Table 1.** Results for the Problem of Figure 1.

<table>
<thead>
<tr>
<th>y</th>
<th>Exact $\Phi$</th>
<th>BEM $\Phi$</th>
</tr>
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<tbody>
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<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
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<td>0.4588</td>
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<td>0.5781</td>
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<tr>
<td>0.5</td>
<td>0.7476</td>
<td>0.7472</td>
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<tr>
<td>0.6</td>
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<td>0.8115</td>
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<tr>
<td>0.7</td>
<td>0.8700</td>
<td>0.8672</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9200</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.9600</td>
<td>0.9603</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

As can be seen, the results match perfectly to three-decimal place accuracy; the deviation of the BEM solution from the exact results past this point is attributed to numerical error due to the discretization.
Figure 2a: Problems 2 and 3 as cast

Figure 2b: The transformed axisymmetric problems

4 Examples 2 & 3

The geometry of these two problems is identical to that of problem 1 (Figure 2a). The potential $\phi$ on sides $a-b$ and $b-c$ is maintained at zero, while that in side $c-d$ is kept at unity. The conductivity is dictated by the function:

$$k(x) = 0.2x + 0.1$$

Problem 2 has the side $d-a$ insulated, while problem 3 maintains the side $d-a$ at zero potential. Both problems lend themselves to an equivalent axisymmetric solution if we select the transformation

$$r = x + 0.5 \quad \text{and} \quad z = y$$

which is depicted in Figure 2b.

Solving the problem analytically is a very tedious exercise in the use of Bessel functions of both the first and second kinds, a fact which may be responsible for this transformation not having been reported in standard sources for manual solution of partial differential equations of this nature. However, the boundary element solution is very simply implemented. The same mesh was used as that for problem 1. Figures 3 and 4 show the results in graphical contour form for a for problems 2 and 3 respectively. The results indicate the obvious qualitatively correct behavior. Because of the tedium of generating an analytical solution with the Bessel series, this was not done.

5 Example 4 --- A New Look at Impermeability

This new approach at solving problems with linearly-varying material properties allows a review of the standard method of managing piecewise nonhomogeneous materials with impermeable or insulated boundaries. In a striated soil, for instance, the problem is normally modelled as a series of 'zones' with different, but individually constant material properties. Impermeable boundaries suddenly
'appear' at the edge of a region of porous material like a discrete layer of plexiglass in a model. Discussions with the second author's colleagues in geotechnical engineering (see Acknowledgements) have indicated that such modelling tactics are done for convenience, mainly because there have not been tractable ways to handle such problems. One would expect a soil's permeability to decrease with depth simply because of the earth pressure. The permeability would only remain effectively constant over relatively shallow depths. Also, in many cases, the state of impermeability would not likely in many cases be a sudden occurrence at the boundary of a soil region. Rather, the onset of impermeability would be a gradual occurrence as the boundary was approached.

This next model illustrates how such a problem could be approached using the axisymmetric transformation. Figure 5a shows another rectangular region
Figure 5a: Problem 4 originally. The hashed hashed lines denote impermeability

Figure 5b: The transformed axisymmetric problem

region of sides $H$ and $L$ with a permeability (conductivity) given by $k(x) = k_1 x$, which is handled with the transformations:

$$r = k_1 x \quad \text{and} \quad z = k_1 y$$

shown in Figure 5b. The boundary conditions are zero potential specified at the top and the bottom of the region, and a potential of unity at the right. Note that the flux $q = -k_1 x \cdot (\partial \Phi / \partial n)$ is zero or impermeable at the left hand side of the region. The transition to impermeability occurs quite gradually as the left-hand side is approached.

The solution obtained by separation of variables and subsequent expansion into Bessel series is

$$\Phi(r, z) = \frac{2\Phi_0}{k_1 L} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r) \sinh(\alpha_n z)}{\alpha_n j_1(\alpha_n k_1 L) \sinh(\alpha_n k_1 H)}$$

where the $\alpha_n$ are successive roots of the equation

$$J_0(\alpha_n k_1 L) = 0$$

Problem 4 was modelled as a square of $H = L = 5$ and with $k_1 = 2$. The transformation to an axisymmetric BEM formulation is shown in Figure 5b. The tedious series was summed to about 100 terms using when it was recognized that no change would be forthcoming by continuing the sum. The BEM solution gives comparable results with considerably less labor, and it is depicted in Figure 6a. The percentage difference between the explicit results and the BEM solution is shown in Figure 6b, and the error is quite small except at one edge, a small region where the series solution was observed to be having trouble converging. Thus the error in that region is attributed to the analytical solution as opposed to the numerical BEM solution.
Figure 4. The RCM results distribution for problem 1 with the experimental results.

Figure 5. The difference between the analytically computed potentials and the RCM results.
6 Summary and Conclusions

A numerical verification of the two-dimensional BEM formulation for potential problems involving linearly varying materials has been presented. Four examples have shown that the boundary element formulation and solution of such problems is a very simple and accurate process if one has an axisymmetric program for solving the Poisson problem. Also, new light has been shed upon the realistic modelling of real modelling of media containing impermeable media.

The BEM method is arguably, for a general case, the simplest technique for solving real problems of this nature. This formulation is a major step forward in BEM technology since linear variations in media can be handled without explicit domain gridding, something that cannot be said for any of the other popular numerical methods in engineering.

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References


3. Ortiz, J.C. Use of the Boundary Element Method for Analysis of Galvanic Corrosion in Nonhomogeneous Electrolytes. Ph.D. Dissertation, Oklahoma State University, May 1989. (All analyses in this paper were performed using the computer program developed in this work.)