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A boundary element solution to the vibration problem of bidimensional structures on a wide frequency range

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Abstract

A direct boundary element method is presented for the dynamic analysis of thin elastic plates and membranes of arbitrary shape. The formulations employ the frequency domain dynamic fundamental solutions of the problems. The fundamental solutions are then approximated by analytic solutions built from the expressions of the asymptotic fields of the exact solutions. The perimeter is discretized by linear boundary elements. Some numerical examples on circular membranes and plates for both free and forced vibrations are presented to prove the accuracy of the formulation.

1 Introduction

Analytical solutions to the problem of the vibrating plate, such as modal analysis are limited to very simple geometry and boundary conditions. Numerical solutions like finite element method (FEM), e.g. Zienkiewicz[1] and boundary element method (BEM), e.g. Toumi[2] can deal with this kind of problem. But if the frequency range of the studied phenomenon increases the classical numerical solutions become rapidly inaccurate because of the increasing time of computing. As far as very high frequencies are concerned, the statistical energy analysis (SEA), e.g. Lyon[3] is able to describe the global energy levels in the different sub-systems of a complicated structure. New solutions have been recently developped in order to caracterise the structural vibrations in the mid frequency range. Among those, one can mention the Energy flow formulation whose aim is to evaluate the average energy density variables in the structures, e.g. Wohlever & Bernhard[4] and Ichchou, Lebot & Jezequel[5].
In this paper, a boundary element approach is presented to treat the dynamic behaviour of membranes and plates subjected to high frequency level harmonic loads. The paper employs the direct conventionnal B.E.M. for the dynamic analysis of thin, elastic, flexural plates and membranes of arbitrary but smooth

planform under any boundary and loading conditions. The corner effects are not taken into account, which simplifies the formulation. The aim of this paper is to exhibit an approximation of the Green function, by using the asymptotical properties of the fundamental solution. This gives the possibility to treat with subsquent time optimisation, any problems of harmonic loadings on a wide frequency range. The effectiveness and the accuracy of the method is demonstrated by presenting several numerical examples for both circular membranes and plates loaded by a harmonic force located at the center of the structure. The results are compared with those available from analytical methods.

2 Integral formulation of the plate

The governing equation of flexural motion of a homogeneous, isotropic, thin and linear elastic plate of surface Ω and smooth perimeter Γ, under the assumption of small deflections, can be written:

$$\nabla^4 w + \frac{\rho h}{D}\ddot{w} = \frac{q}{D}. \tag{1}$$

Where w=w(x,t) is the lateral deflection, ρh is the mass density per unit area, h is the thickness, q=q(x,t) is the lateral load per unit area and $D = Eh^3/12(1-\upsilon^2)$ is the flexural rigidity of the plate, with E and v being the modulus of elasticity and Poisson's ratio, respectively. Assuming harmonic loading and hence of the deflection, eqn. (1) becomes:

$$\nabla^4 \overline{w} - \beta^4 \overline{w} = \frac{\overline{q}}{D}. \tag{2}$$

The symbol $\overline{(-)}$ represents the amplitude of the lateral deflection and the loading.

$$\beta^4 = \frac{\rho h \omega^2}{D} \tag{3}$$

ω denotes the circular frequency of vibration.
The following boundary integral formulation provides from the classical application of the reciprocal theorem between the two unrelated elastodynamics states, represented by the fundamental solution, G and its derivatives associated to eqn(2), and the studied physical solution, e.g. Butterfield & Bannerjee[6]. The first integral equation has the form:

$$\begin{aligned} c w(\xi) = \frac{1}{D}\int_\Gamma \Big[& V_n(G(\beta r))\overline{w}(x) - M_n(G(\beta r))\frac{\partial \overline{w}(x)}{\partial n} + \\ & M_n(\overline{w}(x))\frac{\partial G(\beta r)}{\partial n} - V_n(\overline{w}(x))G(\beta r)\Big] d\Gamma(x) - \frac{1}{D}\int_\Omega q(x)G(\beta r)d\Omega(x) \end{aligned} \tag{4}$$

where c=1 for $\xi \in \Omega$, and c=1/2 for $\xi \in \Gamma$, and $r = |x - \xi|$. V_n, M_n, $\partial/\partial n$, and n represent the shear force, normal bending moment, normal slope and outward vector, respectively. The second boundary integral equation can be obtained from Eq. (4) by taking its directional derivative $\partial/\partial x$. Thus, one can obtain for the case of $\xi \in \Gamma$ (c=1/2):

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$$\frac{1}{2}\frac{\partial w(\xi)}{\partial x}=\frac{1}{D}\int_{\Gamma}\left[\frac{\partial V_n(G(\beta r))}{\partial x}\overline{w}(x)-\frac{\partial M_n(G(\beta r))}{\partial x}\frac{\partial \overline{w}(x)}{\partial n}\right.$$
$$\left.+M_n(\overline{w}(x))\frac{\partial^2 G(\beta r)}{\partial n \partial x}-V_n(\overline{w}(x))\frac{\partial G(\beta r)}{\partial x}\right]d\Gamma(x)-\frac{1}{D}\int_{\Omega}q(x)\frac{\partial G(\beta r)}{\partial x}d\Omega(x) \quad . (5)$$

The numerical solution of eqns (4) and (5), will be accomplished by discretising the boundary and by writing the above equations in discrete form.

3 Integral formulation of the membrane

The integral formulation of the membrane can be developped in the same manner as it has been done for the plate. The considered membrane is subjected to a harmonic load. The governing equation of a homogeneous, isotropic, and linear elastic membrane with the assumption of small deflections under in-plane tensile force T, has the form:

$$\nabla^2 w-\frac{\rho}{T}\ddot{w}=-\frac{q}{T}. \quad (6)$$

Where w is the lateral deflection, ρ is the mass density per unit area. q denotes the lateral loading of the membrane per unit area. In the assumption of harmonic loading and hence of the deflection eqn (6) becomes:

$$\nabla^2\overline{w}+\frac{\rho\omega^2}{T}\overline{w}=-\frac{\overline{q}}{T}. \quad (7)$$

ω is the singular frequency and the symbol (-) denotes amplitudes. The application of the reciprocal theorem leads to the boundary integral equation, where G is the fundamental solution associated to eqn (7):

$$\frac{1}{2}\overline{w}(\xi)=\int_{\Omega}\overline{q}(x)G(\beta r)d\Omega(x)+\int_{\Gamma}G(\beta r)V_n(\overline{w}(x))-\overline{w}(x)V_n(G(\beta r))d\Gamma(x). \quad (8)$$

As well as for the plate, the numerical solution of eqn (8) will be carried out by discretising the boundary and by writing the above equation in discrete form.

4 The approximate kernel

In this section we build an approximation of the fundamental solution that will prevent from going ahead to numerical problems when evaluating the Green solution at high frequency levels. The major idea of this work relies on the fact that one can replace the exact solution, build with Hankel functions, by an asymptotic solution. We consider a near field and a far field and we must set up a transition between the two fields. The approximate solution is supposed to have the same properties than the exact one. That is to say, the new built solution and its required derivatives must be continous.

Both for the plate and the membrane, the two fundamental solutions and there derivatives are a combination of Hankel functions of the first kind of zero and first order. These functions are expressed by series whose asymptotic fields have been numerically settled by Abramovitz[7], they are reminded in appendix A. Therefore, the transition functions will be realized between the near and far fields of the Hankel functions. From this starting point we propose a method to

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create the approximate field H(βr). The near field (respectively the far field) is called $F_n(\beta r)$ (respectively $F_f(\beta r)$). r is the radial coordinate.
Two points are considered, a and b, $a \leq b$

$$\begin{gathered} \forall \beta r \leq a \quad \mathrm{H}(\beta r) = F_n(\beta r) \\ \forall \beta r \geq b \quad \mathrm{H}(\beta r) = F_f(\beta r) \\ \forall \beta r \in [a,b] \ \mathrm{H}(\beta r) = F_f(\beta r).f(\beta r) + F_n(\beta r).n(\beta r) \end{gathered} \tag{9}$$

The aim is to choose the functions f and n so that

$$\begin{gathered} F_n(a) = F_n(a).n(a) + F_f(a).f(a) \\ F_f(b) = F_n(b).n(b) + F_f(b).f(b) \end{gathered} \tag{10}$$

f and n must satisfy, $n(a) = f(b) = 0$, $n(b) = f(a) = 1$. Our goal is to build smooth fundamental solutions. The functions n and f will be chosen so that $\forall x \succ 0 \quad H(x) \in C_\infty$. A very simple calculus shows that this property requires the following condition on the f and n functions

$$\forall n \in N^* \quad f^{(n)}(a) = f^{(n)}(b) = n^{(n)}(a) = n^{(n)}(b) = 0 \tag{11}$$

One can find a lot of different functions verifying the enounced properties. The expression of the proposed transition function is:

$$\begin{gathered} \forall\, \mathrm{x} \in \left[\frac{\mathrm{b+a}}{2}, b\right[, \quad f(x) = \frac{1}{2}\left(1 + \sqrt{1 - e^{-\frac{1}{2}\left[\tan\left(\frac{\pi}{b-a}\left(x - \frac{1}{2}(b+a)\right)\right)\right]^2}}\right) \\ \forall\, \mathrm{x} \in \left]a, \frac{\mathrm{b+a}}{2}\right], \quad f(x) = 1 - f(a+b-x) \\ f(a) = 0,\ f(b) = 1 \end{gathered} \tag{12}$$

One can easily prove the function f is C_∞ on $[a,b]$, refering to the derivative prolongation theorem, e.g. Lelong-Ferrand & Arnaudiès[8], and the values of the successive derivatives at the boundary of the interval are all equal to zero. The approximate fundamental solution can now be writen

$$\begin{gathered} \forall\ 0 \leq \beta r \leq \mathrm{a},\ \mathrm{H}(\beta r) = F_n(\beta r), \quad \forall\ \beta r \geq \mathrm{b},\ \mathrm{H}(\beta r) = F_f(\beta r) \\ \forall\ \beta r \in]a,b[, \quad \mathrm{H}(\beta r) = F_n(\beta r).\{1 - f(\beta r)\} + F_f(\beta r).f(\beta r) \end{gathered} \tag{13}$$

Eqn (13) provides an analytic formulation of the approximate fundamental solution. a and b are non dimensionnal constants fixed by the expressions of the far field and the near field. In Figure 1a, a comparison between the exact and the approximate fundamental solution for the membrane is presented. In Figure 1b the same comparison for the plate is presented.

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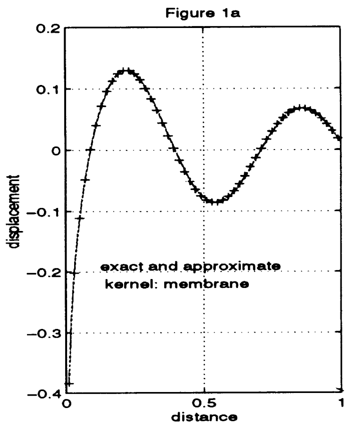

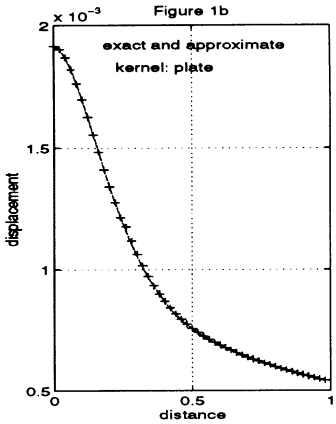

5 Numerical examples

Example 1
Consider a clamped circular membrane of radius R=1m, mass per unit volume ρ=10kg/m^2, and tensile force T=10 and ν=0.3, subjected to a concentrated vertical harmonic loading at its center with magnitude F_0=1. A hysteretic damping ratio of two percent is added. Five boundary elements are used, with linear interpolation functions. The method is compared with an "exact" analytic solution developped by Graff[9], as shown in Figure 2.

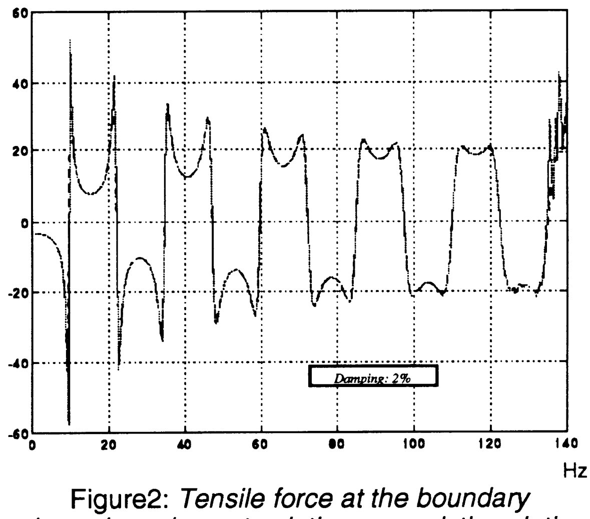

Figure2: *Tensile force at the boundary*
--boundary element solution — analytic solution

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Example 2
Consider a clamped circular plate of radius R=0.5, thickness h=0,001, mass per unit surface ρ=7800kg/m^3, and elastic constants E=2,10^{11} and ν=0.3, experiencing free vibrations. Those are obtained by a method developped by Kitahara[10]. Table 1 presents the first three symmetric natural frequencies. The obtained results are compared to the "exact" solution developped by Graff[9], and to a finite element software, ANSYS. Once again, five boundary elements with linear interpolation functions are used.

Hz	Viktorovitch & al	ANSYS	Graff
f1	10,07	10,20	9,97
f2	38,91	39,63	38,80
f3	86,93	88,87	86,92

Table 1: natural frequencies of the first three symmetric modes

6 Conclusion

A direct boundary element formulation, for the dynamic analysis of thin elastic plates and membranes of arbitrary shape, has been described. The formulations employ the frequency domain dynamic fundamental solutions of the problems. The main conclusions that can be drawn from the previous discussion are: (a) the approximate fundamental solution brings subsequent time saving on the solution; (b) the evaluation of the approximate kernel circumvents the computational difficulties arising from the use of the dynamic fundamental solution (c) plates and membranes of arbitrary shape subjected to any kind of loading and boundary conditions can be considered; (d) The accuracy of the results is proved on a wide frequency range for both free and forced vibrations.

References

1. O.C. Zienkiewicz & R.L. Taylor, *The finite element method*, fourth edition, Mc Graw-Hill, London, 1989.

2. K. Toumi, *Analyse statique et dynamique de structures mécano-soudées par la méthode des équations intégrales de frontière*, thèse de mécanique Ecole Centrale de Lyon, 1994.

3. R.H. Lyon, *Statistical Energy Analysis of dynamical systems: theory and application*, MIT Press, Cambridge, 1975.

4. J.C. Wohlever & R.G. Bernhard, *Mechanical energy models for rods and beams*, Journal of Sound and Vibration, 1992, **153(1)**, 1-19.

5. M.N. Ichchou, A. Lebot & L.Jezequel, *Radial and tangential energy flow models for curved wave guides*, Proceeding of the Vibration and noise conference '95, Venice, Italy, April 1995.

6. R. Butterfield. & K. Bannerjee, *Boundary element methods in engineering science*, Mac Graw-Hill, New-York, 1981.

Transactions on Modelling and Simulation vol 10, © 1995 WIT Press, www.witpress.com, ISSN 1743-355X

7. M. Abramovitz, *'Hand book of mathematical functions'*, Springer Verlag, Berlin, 1983.

8. J. Lelong-Ferrand & J.M. Arnaudiès, *Cours de mathématiques, Analyse,* Tome 2, 4eme édition, Dunod Université, 1977.

9. K. F. Graff, *Wave motion in elastic solids*, Dover publications, New-York, 1991.

10. M. Kitahara, *'Boundary integral equation methods in eigenvalue problems of elastodynamics and thin plates'*, Elsevier, Amsterdam, 1985.

Appendix A

Expressions of the near field of Bessel and Neumann functions of the first kind of zero and first order:

$$0 \leq x \leq 3$$

$$J_0(x) = 1 - 2.2499997(x/3)^2 + 1.2656208(x/3)^4 - 0.3163866(x/3)^6$$
$$+0.0444479(x/3)^8 - 0.0039444(x/3)^{10} + 0.0002100(x/3)^{12} + \varepsilon$$
$$|\varepsilon| \leq 5,10-8$$

$$Y_0(x) = (2/\pi)\ln\left(\frac{1}{2}x\right)J_0(x) + 0.36746691 + 0.60559366(x/3)^2$$
$$-0.74350384(x/3)^4 + 0.25300117(x/3)^6$$
$$-0.04261214(x/3)^8 + 0.00427916(x/3)^{10} -$$
$$0.00024846(x/3)^{12} + \varepsilon$$
$$|\varepsilon| \leq 1.4,10^{-8}$$

$$x^{-1}J_1(x) = \frac{1}{2} - 0.56249985(x/3)^2 + 0.21093573(x/3)^4 - 0.03954289(x/3)^6$$
$$+0.00443319(x/3)^8 - 0.00031761(x/3)^{10} + 0.00001109(x/3)^{12} + \varepsilon$$
$$|\varepsilon| \leq 1.3,10^{-8}$$

$$xY_1(x) = (2/\pi)x\ln\left(\frac{1}{2}x\right)J_1(x) - 0.6366198 + 0.2212091(x/3)^2$$
$$+2.1682709(x/3)^4 - 1.3164827(x/3)^6 + 0.3123951(x/3)^8$$
$$-0.00400976(x/3)^{10} + 0.0027873(x/3)^{12} + \varepsilon$$
$$|\varepsilon| \leq 1.1,10^{-7}$$

Expression of the far field of the Hankel function of the first kind of zero and first order:

$$3 \le x \le \infty$$

$$J_0(x) = x^{-\frac{1}{2}} f_0 \cos\theta_0 \quad Y_0 = x^{-\frac{1}{2}} f_0 \sin\theta_0$$

$$f_0 = 0.79788456 - 0.00000077(3/x) - 0.00552(3/x)^2$$
$$-0.00009512(3/x)^3 + 0.00137237(3/x)^4 - 0.00072805(3/x)^5$$
$$+0.00014476(3/x)^6 + \varepsilon$$
$$|\varepsilon| \le 1.6{,}10^{-8}$$

$$\theta_0 = x - 0.78539816 - 0.04166397(3/x) - 0.00003954(3/x)^2$$
$$+0.00262573(3/x)^3 - 0.00054125(3/x)^4 -$$
$$0.00029333(3/x)^5 + 0.00013558(3/x)^6 + \varepsilon$$
$$|\varepsilon| \le 7{,}10^{-8}$$

$$J_1(x) = x^{-\frac{1}{2}} f_1 \cos\theta_1 \quad Y_1(x) = x^{-\frac{1}{2}} f_1 \sin\theta_1$$

$$f_1 = 0.79788456 + 0.00000156(3/x) + 0.01659667(3/x)^2$$
$$+0.00017105(3/x)^3 - 0.00249511(3/x)^4 +$$
$$0.00113653(3/x)^5 - 0.00020033(3/x)^6 + \varepsilon$$
$$|\varepsilon| \le 4{,}10^{-8}$$

$$\theta_1 = x - 2.35619449 + 0.12499612(3/x) + 0.00005650(3/x)^2$$
$$-0.00637879(3/x)^3 + 0.00074348(3/x)^4 + 0.00079824(3/x)^5$$
$$-0.00029166(3/x)^6 + \varepsilon$$
$$|\varepsilon| \le 9{,}10^{-8}$$