The local assembled form of MLGFM
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Abstract
The Modified Local Green's Function Method, MLGFM, has been proposed as an alternative for extending the applicability of the Boundary Element Method to problems which do not have a known fundamental solution. Up to now, most of the published results are based on a global version where only one Green's Cell is used. In this paper, some results for nonlinear potential problems obtained by using several Green's Cells are reported and compared against Mixed Finite Element solutions. This comparison illustrates the beneficial effects that the boundary elements bring in the formulation: improved accuracy and fewer iterations in nonlinear problems.

Introduction
Although the Boundary Element Method, BEM, has long been recognized as an established numerical method with many advantages over other techniques, its formulation may become cumbersome when a fundamental solution is not available or has a too complicated form. The Modified Local Green's Function Method, MLGFM, was proposed by de Barcellos & Silva [1] as an alternative to extend the BEM methodologies to problems which have no known fundamental solution in its explicit form or are very intricate. Some examples of these are
potential problems in non-homogeneous media, deformation of plates with variable thickness and shells of arbitrary shape.

Most of the already published work, see e.g. [2,3,4], on the MLGFM have made use of the global version, just one Green's cell, where the accuracy and convergence properties have been investigated. These previous work have laid down the understandings necessary to implement efficient codes for multiple cell models. Essentially, the MLGFM uses a very coarse mesh of Green's cells where each one of them requires a coarse finite element, FE, mesh and associated boundary element, BE, mesh in order to locally evaluate the Green's function projections on the FE and BE interpolations functions. After these projections are computed, they are introduced in the direct BE equations and the Galerkin-BEM formalism is applied. The boundary values and continuity requirements between the cells are introduced as in BEM and the boundary cell unknowns are evaluated. If the values of the unknown function inside a cell are desired, one may again, for higher accuracy, use the MLGFM in such a cell, now designated as macrocells, by using its solution as boundary conditions for the microcell's mesh into which it is divided. In doing so, one is using the same sort of ideas as domain partition and multi-grid concepts.

Although not shown explicitly in this paper, two major advancements have recently been developed and used in conjunction with the formulation just described. Firstly, it was obtained a fast way to compute the Green's Functions Projections over a single element cell without performing any matrix inversion or product. Secondly, a fast iterative procedure was developed to compute the secondary variables at internal nodes on each cell, taking the values at its boundary computed at the first step of processing, as boundary values. To differentiate the procedures, this one is called Iterative Local Green, ILG, in opposition to the Assembled Local Green, ALG, briefly reviewed in the paragraph above.

Modified Local Green's Function

Consider the differential equation for a non-homogenous and nonlinear potential problem
\[ -\nabla \cdot Z(u,P) \nabla u = b(P), \quad P \in \Omega \]
\[ u(P) = \overline{u}(P), \quad p \in \Gamma_u \]
\[ -\hat{n} \cdot Z \nabla u = f(p) = \overline{f}(p), \quad p \in \Gamma_f \tag{1} \]

where: \( \Omega \), \( \Gamma_u \) and \( \Gamma_f \) stand for the domain and boundary partitions where the Dirichlet, \( \overline{u} \), and Neumann, \( \overline{f} \), boundary conditions are specified; \( Z(u,P) \) is a non-homogeneous material property which might be also function of the potential \( u \); and \( b \) is the excitation function. Initially, the domain is partitioned in a coarse mesh of Green's cells. Next, consider for each cell the integral equation

\[ u(Q) = \int_{\Omega} G(P,Q) b(P) d\Omega + \frac{1}{\Omega} \int_{\Gamma} [u_u + \alpha(p) u(p)] G(p,Q) d\Gamma \]
\[ -\frac{1}{\Omega} \int_{\Gamma} [G_u + \alpha(p) G(p,Q)] u d\Gamma \tag{2} \]

where \( \alpha \) is a convenient scalar operator and \( G \) is the Green's function of the First Auxiliary Problem:

\[ -\nabla^2 G(P,Q) = \delta(P,Q) \quad \forall P, Q \in \Omega \]
\[ G_{\alpha} + \alpha(p) G(p,Q) = 0 \quad \forall p \in \Gamma \tag{3} \]

Defining \( F(p) = u_u + \alpha(p) u(p) \) and considering (3), one can rewrite (2) as

\[ u(Q) = \int_{\Omega} G(P,Q) b(P) d\Omega + \frac{1}{\Omega} \int_{\Gamma} F(p) G(p,Q) d\Gamma, \quad \forall Q \in \Omega, \ p \in \Gamma \tag{4} \]

Define the Second Auxiliary Problem

\[ -\nabla^2 G(p,q) = 0 \quad \forall P \in \Omega, \ q \in \Gamma \]
\[ G_{\alpha} + \alpha(p) G(p,q) = \delta(p,q) \quad \forall p, q \in \Gamma \tag{5} \]

Using (5) and taking the limit in (2) as the inner point "\( Q \)" approaches a boundary point "\( q \)" one obtains

\[ u(q) = \int_{\Omega} G(P,q) b(P) d\Omega + \frac{1}{\Omega} \int_{\Gamma} F(p,q) G(p,q) d\Gamma, \forall P \in \Omega, \ p, q \in \Gamma \tag{6} \]

One can notice that equations (2) and (6) are the same equations used in BEM except for the variable \( \alpha \) and that \( G \) now stands for the Green's function associated with the two auxiliary problems. The next step is to use the Galerkin
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Method on the equations (2) and (6). For this purpose, one defines a set of FE interpolations, \( \{ \psi_j \} \), defined on each domain partition (cell), and a set of BE interpolation functions, \( \{ \phi_j \} \), such that each \( \phi_j \) is the trace of a \( \psi_j \). As a result of the Galerkin procedure, one obtains two sets of algebraic equations, namely

\[
Au^d = BF^e + Cb^d \\
Du^c = EF^e + Fb^d
\]

where \( u^d \) and \( u^c \) are the nodal domain and boundary values of \( u \), respectively, and the matrices are defined as

\[
A = \int_{\Omega} [\psi(Q)]^t [\psi(Q)] d\Omega_Q \\
B = \frac{\partial}{\partial \Gamma} \int_{\Gamma} [\phi(q)]^t [\phi(q)] d\Gamma_q \\
C = \int_{\Omega} [G_d(P)]^t [\psi(P)] d\Omega_P \\
D = \frac{\partial}{\partial \Gamma} \int_{\Gamma} [G_c(p)]^t [\phi(p)] d\Gamma_p
\]

\[
F = \int_{\Omega} [G_c(p)]^t [\psi(P)] d\Omega_P
\]

The Green's function projections, \( G_d \) and \( G_c \), are obtained by projecting the quantities in equations (3) and (5) over the subspaces generated by \( \{ \psi_j \} \) and \( \{ \phi_j \} \), respectively, and then solving the resulting auxiliary problems:

\[
-\nabla_p^2 G_d(P) = [\psi(P)]_p, \quad \forall P \in \Omega \\
G_{d,n}(p) + \alpha(p)G_d(p) = 0, \quad \forall p \in \Gamma
\]

\[
-\nabla_p^2 G_c(P) = 0, \quad \forall P \in \Omega \\
G_{c,n}(p) + \alpha(p)G_c(p) = [\phi(p)]_p, \quad \forall p \in \Gamma
\]

In order to obtain the compatibility between the Green's cells one proceeds as in BEM, that is, one requires the potential and flux continuities. In the case pictured in Fig. 1, one requires \( u_{12}^i = u_{11}^e \) and \( f_{12} = -f_{11} \).

One can devise two schemes for solving the resultant system of equations: to construct a global matrix equation to be solved once in linear problems, or to formulate an iterative procedure. The latter has shown some sensitivity of the convergence rate on \( \alpha \) in nonlinear problems. Here the value of \( \alpha \) has been taken as a constant for each Green's cell and the iterative procedure...
was chosen and procedures for greatly reducing the computational effort have proved to be effective.

In nonlinear problems, one takes a reference value for $Z$ to be used in the left hand side of the differential equation and the term which contains the difference between the actual and the reference value is moved to the right hand side and added to $b$. So, in nonlinear problems, the nodal vector $b^e$ in eqs. (7) are also functions of $u^e$. Hence, one has to solve for them iteratively. The factor $\alpha$ has been shown to influence the matrix conditioning as well as the convergence. For now, $\alpha$ has been selected as a constant, $c_{nd}$, for each cell according to the following estimator $c_{nd} = \frac{\sum_i |K_{n,i}|}{\sum_j |K_{j,i}|}$ which has shown to be near the optimum in a low sensitivity neighborhood.

**Numerical Examples**

Find an approximate solution for the problem (1) with $Z(u,P) = a_u + a_i u$, $\Gamma_f = \emptyset$ and $\bar{u}(p) = 0$ over the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : -1 < x, y < 1\}$. Due to symmetries, only one fourth of the domain is modeled. In order to compare the results, a Mixed FE code was developed using the same iterative schemes as in MLGFM. The material constant $a_u$ was set equal to 1.0 and $a_i$ equal to 0.1 and 1.0. Figure 2 shows how appropriate the procedure for estimating $c_{nd}$ has shown to be. The relative errors in flux, $E_f$, in potential, $E_u$, and the residual error $R_r$, are defined as

$$E_f = \frac{|u - u_{shf(\Omega)}|^2}{|u_{shf(\Omega)}|^2} \quad E_u = \frac{|u - u_{sh2(\Omega)}|^2}{|u_{sh2(\Omega)}|^2} \quad R_r = \left|\frac{R_{N,NTN} - R_{N,NTN-1}}{|R_{N,NTN}|}\right|^2$$

(11)
where $R$ and $R_N$ are the linear and nonlinear parts of the excitation vector and NITN is the index of the nonlinear iteration. Here, the number of iterations locally performed in each cell is restricted to three and the iterations in the nonlinear loop are stopped when $Rr$ becomes less than $1.0e-10$. In Figures 3, 4
and 5, the comparison of the MLGFM against the FEM results are shown. It is worth noting that the relative errors for the potential decreases faster than for the flux and, moreover, the relative residual error is smaller than that of FEM and the rate of convergence is of the order of 9/4 as compared to 9/6 for the FEM.

Figure 4. Error $E_u$ of the potential for 4x4 biquadratic macrocells of 3x3 mcl.

Conclusions

The MLGFM scheme for solving nonlinear problems using its local version has briefly been outlined. Among the first problems solved, typical one is presented which illustrates the convergence properties of MLGFM as compared to a mixed FE formulation which is the one in the FEM family which most resembles the MLGFM.

We have verified that boundary elements increase the accuracy and the rate of convergence, even by using FEM to compute the Green's function projections. These results suggest its use for other nonlinear continuum mechanics problems such as fluid flow and plasticity.

As mentioned, the new implementations have substantially reduced the computational effort, resulting in a technique favorably competitive with both BEM and FEM.
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Figure 5. Relative Residue $R_r$ for 4x4 biquadratic macrocells of 3x3 mcl.

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