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Design sensitivity analysis of three-dimensional elastostatic solids using the boundary integral equation method

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Abstract

Design sensitivity of three-dimensional structures using the Boundary Integral Equation Method (BIEM) is investigated. Direct implicit differentiation of BIEM equations is performed analytically to obtain traction, displacement and equivalent stress sensitivities for the design optimization. A new procedure is employed - instead of differentiating equations with respect to co-ordinates of the design variable, they are differentiated with respect to the normal to the surface at the design variable point. Hence the computational time is reduced by a factor of three. During the numerical evaluation of analytically derived BIEM sensitivity equations, triangular and quadrilateral elements are used. Some nodes of a movable part of the body boundary are chosen as the design variables. The rigid body translation method is used to remove singular integrals from the equations. Two test problems: a cube, and a hole in a cube under biaxial loading are demonstrated. Comparison of the sensitivity results is made with those using a Finite Difference Method (FDM), and excellent accuracy is achieved.

1. Introduction

Sensitivity analysis is one of the most important aspects in the structural optimization method. Design sensitivity refers to computing the change of an object's behaviour with respect to changes in the design variables that affect its shape. The need for a more general and applicable method for design sensitivity analysis has become more important as optimization methods becomes increasingly popular. Sensitivity analysis problems are mainly investigated by two different methods. The traditional method was to simply change the geometry of the domain by giving a very small movement to a design variable, reanalyze the structure, and compare the two calculated results of structural response. Sensitivities of static structures are investigated with the design variation method. The effect of design variation on the properties of a structure is considered using a variational formulation for derivatives of eigenvalues with respect to design variables [1]. The boundary of the structure to be optimized is represented by a set of shape functions. Then the shape optimization procedure is reduced to the determination of paramaters such as thickness and width or other parameters [2]. An adjoint variable method is used, coupling with the material derivatives of continuum mechanics to determine the sensitivity of structures [3]. The variational formulation method is mainly applied to structural problems using the finite element method. It is quite difficult to apply the finite element method straightforwardly because of the complex nature of finite difference equations, and mesh generation causes extra problems [4]. Using the same variation derivation concept which is discussed above, the BIEM is employed instead of the finite element method since the BIEM is known to be more accurate in determining stresses. A general formulation for stress sensitivities is derived by using the BIEM and the effect of the choice of the adjoint load is discussed in [5]. In the second method, sensitivity analysis equations are obtained by direct differentiation of equations. This method involves implicit differentiation of the basic equation of structural response. Finite difference schemes using the implicit differentiation method are used in many structural optimization problems [9, 10]. The BIEM can analyze the boundary values more precisely than the finite element analysis. But, system matrixes are unsymmetric and fully populated. Hence a good sensitivity procedure is required to solve large scale problems during optimization. The LU decomposition method allows the solution of big problems efficiently and less computational time is required than gaussian elimination method. Hence, the implicit differentiation method appears best suited to the boundary element method [7]. Direct differentiation of BIEM is employed in many problems [6, 11]. More general information on the comparison between the variational and implicit differentiation method can be found in [8].

2 Review of the boundary integral equation method

With the aid of Betti's reciprocal theorem, Somigliana's identity for the displacement at any interior point p can be expressed in terms of integrals over the boundary (Q is a point on the boundary) as

$$u_i(p) = \int_S t_j(Q) U_{ij}(p,Q)dS - \int_S u_j(Q) T_{ij}(p,Q)dS \tag{1}$$

Where $U_{ij}(p,Q)$ and $T_{ij}(p,Q)$ are the displacement and traction kernel functions associated with Kelvin's solution for a point load in an infinite medium [12].

$$U_{ij}(p,Q) = \frac{1}{16\pi\mu(1-\upsilon)}\left(\frac{1}{r}\right)\left[(3-4\upsilon)\delta_{ij} + r_{,i}r_{,j}\right] \tag{2}$$

and

$$T_{ij}(p,Q) = -\frac{1}{8\pi(1-\upsilon)}\left(\frac{1}{r^2}\right)\left\{\frac{\partial r}{\partial n}\left[(1-2\upsilon)\delta_{ij} + 3r_{,i}r_{,j}\right] - (1-2\upsilon)\left(n_j r_{,i} - n_i r_{,j}\right)\right\} \tag{3}$$

Using appropriate limiting techniques [13], p can be taken to the boundary as point P, to give

$$C_{ij}(P)u_j(P)j + \int_S T_{ij}(P,Q)u_j(Q)ds = \int_S U_{ij}(P,Q)t_j(Q)ds \tag{4}$$

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2.1 Numerical formulation of the BIEM

The domain is divided into triangle and quadrilateral elements having six or eight nodes. The boundary geometries of these elements are represented with intrinsic co-ordinates and quadratic shape functions $N(\xi)$. The Jacobian of transformation from local intrinsic co-ordinates to global co-ordinate system can be expressed by

$$|J(\xi)| = \left(J_1^2 + J_2^2 + J_3^2\right)^{1/2} \tag{5}$$

where, J_1, J_2, and J_3 are the components of the Jacobian [12]. The unit outward normal is given by

$$n_i = J_i / |J(\xi)| \qquad i = 1,3 \tag{6}$$

Let $d(b,c)$ be the c'th node of the b'th element, where $c = 1, C$, a quadrilateral or triangular element's node number and $b = 1, M$. M is the total number of elements. Eq. (4) can be expressed in the intrinsic co-ordinate system.

$$C_{ij}(P_n)u_j(P_n) + \sum_{b=1}^{M}\sum_{c=1}^{C} u_j\left(P_{d(b,c)}\right)\int_{S_b} T_{ij}\left(P_n, Q(\xi)\right)N^c(\xi)|J(\xi)|d\xi =$$
$$\sum_{b=1}^{M}\sum_{c=1}^{C} t_j\left(P_{d(b,c)}\right)\int_{S_b} U_{ij}\left(P_n, Q(\xi)\right)N^c(\xi)|J(\xi)|d\xi \tag{7}$$

Eq. (7) represents a 3N linear equation system with 6N nodal point values of displacements and tractions where half of them are known. N is the total nuber of nodal points. The problem can be reduced to

$$A\overline{U} = B \tag{8}$$

where the A and B matrices contain traction and displacement kernels which are calculated numerically, and $\overline{U}$ is the vector of unknowns.

3 Differentiation of BIEM system of equations with respect to the normal at a design variable point

The displacement gradients are calculated directly from differentiation of Eq. (8) with respect to the normal (n_k) to the surface at the design variable point.

$$\frac{\partial}{\partial n_k}A\overline{U} + A\frac{\partial}{\partial n_k}\overline{U} = \frac{\partial}{\partial n_k}B \tag{9}$$

Similarly Eq. (7) can be differentiated

$$C_{ij}(P_n)\frac{\partial u_j(P_n)}{\partial n_k} + \sum_{b=1}^{M}\sum_{c=1}^{C}\frac{\partial u_j(P_{d(b,c)})}{\partial n_k}\int_{S_b} T_{ij}(P_n, Q(\xi))N^c(\xi)|J(\xi)|d\xi =$$

$$\sum_{b=1}^{M}\sum_{c=1}^{C}\frac{\partial t_j(P_{d(b,c)})}{\partial n_k}\int_{S_b} U_{ij}(P_n, Q(\xi))N^c(\xi)|J(\xi)|d\xi +$$

$$\sum_{b=1}^{M}\sum_{c=1}^{C} t_j(P_{d(b,c)})\int_{S_b}\frac{\partial U_{ij}(P_n, Q(\xi))}{\partial n_k}N^c(\xi)|J(\xi)|d\xi +$$

$$\sum_{b=1}^{M}\sum_{c=1}^{C} t_j(P_{d(b,c)})\int_{S_b} U_{ij}(P_n,Q(\xi))N^c(\xi)\frac{\partial|J(\xi)|}{\partial n_k}d\xi -$$
$$\sum_{b=1}^{M}\sum_{c=1}^{C} u_j(P_{d(b,c)})\int_{S_b}\frac{\partial T_{ij}(P_n,Q(\xi))}{\partial n_k}N^c(\xi)|J(\xi)|d\xi - \tag{10}$$
$$\sum_{b=1}^{M}\sum_{c=1}^{C} u_j(P_{d(b,c)})\int_{S_b} T_{ij}(P_n,Q(\xi))N^c(\xi)\frac{\partial|J(\xi)|}{\partial n_k}d\xi$$

After knowns and unknowns are arranged as Eq. (9). Our new equation becomes

$$A\overline{U} = B' \tag{11}$$

where B' contains derivatives of kernels with respect to the normal at design variable points.

3.1 Derivatives of traction and displacement kernels with respect to normal at a design variable

During the derivation process three general cases must be considered. Where
1) Load point is the design variable point, field point is outside the adjacent element region.
2) Load point is not the design variable point, field point is inside the adjacent element region.
3) Load point is the design variable ponit and field point is inside the adjacent element region.
Derivatives of the traction kernel are

$$\frac{\partial T_{ij}(P,Q)}{\partial n_k} = -\frac{1}{4\pi(1-\upsilon)}\left(\frac{1}{r^3}\right)\frac{\partial r}{\partial n_k}\left\{\frac{\partial r}{\partial n}\left[(1-2\upsilon)\delta_{ij} + 3r_{,i}r_{,j}\right] - (1-2\upsilon)\right.$$
$$\left.(n_j r_{,i} - n_i r_{,j})\right\} - \frac{1}{8\pi(1-\upsilon)}\left(\frac{1}{r^2}\right)\left\{\frac{\partial}{\partial n_k}\left(\frac{\partial r}{\partial n}\right)\left[(1-2\upsilon)\delta_{ij} + 3r_{,i}r_{,j}\right]\right\} \tag{12}$$

Derivatives of the displacement kernel are

$$\frac{\partial U_{ij}(P,Q)}{\partial n_k} = \frac{1}{16\pi\mu(1-\upsilon)}\left(-\frac{\partial r}{\partial n_k}\right)\frac{1}{r^2}\left[(3-4\upsilon)\delta_{ij} + r_{,i}r_{,j}\right]$$
$$+\frac{1}{16\pi\mu(1-\upsilon)}\left(\frac{1}{r}\right)\left[\frac{\partial r_{,i}}{\partial n_k}r_{,j} + r_{,i}\frac{\partial r_{,j}}{\partial n_k}\right] \tag{13}$$

Derivatives of the jacobian of transformation can be calculated as

$$\frac{\partial|J(\xi)|}{n_k} = \left[\frac{\partial J_1}{\partial n_k}J_1 + \frac{\partial J_2}{\partial n_k}J_2 + \frac{\partial J_3}{\partial n_k}J_3\right]\Big/ J \tag{14}$$

and the unit outward normal's derivative is

$$\frac{\partial n_i}{\partial n_k} = \frac{\left(\frac{\partial J_i}{\partial n_k} \left| J(\xi) \right| - J_i \frac{\partial \left| J(\xi) \right|}{\partial n_k} \right)}{\left| J(\xi) \right|^2} \tag{15}$$

3.2 Derivatives of equivalent stresses

One commonly used equivalent stress criterion is [13];

$$\sigma_e = \frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \tag{16}$$

Where σ_1, σ_2, and σ_3 are the principal stresses. Differentiation of Eq. (16) with respect to the normal at the design variable point is

$$\frac{\partial \sigma_e}{\partial n_k} = \left[(\sigma_1 - \sigma_2)\frac{\partial(\sigma_1 - \sigma_2)}{\partial n_k} + (\sigma_2 - \sigma_3)\frac{\partial(\sigma_2 - \sigma_3)}{\partial n_k} + (\sigma_3 - \sigma_1)\frac{\partial(\sigma_3 - \sigma_1)}{\partial n_k} \right]\sigma^{-1/2} \tag{17}$$

4 Numerical results

In order to test the accuracy of direct differentiation of the BIEM equations, two examples are considered. As a basic example a cube, and as a second example a hole in a cube under biaxial loads are solved.

4.1 Sensitivity analysis of a cube under biaxial stresses

This example contains 6 quadrilateral elements, and 20 nodes. The length of each side of the cube is taken as 5 cm. During the sensitivity analysis the cube is constrained with 3 of its sides as planes of symmetry (those not containing point B in Figure 1). Node B is chosen as the design variable point (Figure 1). The elasticity modulus is equal to 1, Poisson's ratio is taken as 0.3. The loading ratio is $\sigma_1/\sigma_2 = 2$. Where σ_1 is tensile stress acting on face B, C, G, and H, σ_2 is tensile stress acting on face A, B, E, and H. The results are compared with a finite difference method. The increment is equal to $D = 10^{-7}$ cm. Traction, displacement, and equivalent stress derivative results for direct and finite difference methods are shown in Tables 1 and 2. The finite difference and analytical results are almost identical.

4.2 Sensitivity analysis of a cube with a hole under biaxial stresses

Similar examples are often solved in sensitivity analysis problems. Each side of cube is 15 cm. The initial shape of hole is considered as a cylinder with diameter 5 cm. Node L is the design variable (Figure 2). Again the loading ratio is $\sigma_1/\sigma_2 = 2$. Where σ_1 is tensile stress acting on face U, V, S, and T, σ_2 is tensile stress acting on face X, U, V, and Z. Planes of symmetry are applied to consider only one eighth of the problem. It is constrained from three sides. The system contains 51 triangular and quadrilateral elements and 143 nodes. The elasticity modulus and Poisson's ratio are taken as 1 and 0.3.

The increment of position is equal to $D = 10^{-7}$ cm. Again the traction, displacement and equivalent stress results are shown in Tables 3 and 4. Very good agreement is achieved between the finite difference and the analytical differentiation method.

Nodal points	Tractn in X direct. for FD	Tractn. in Y direct for FD	Tractn. in Z direct. for FD	equiv. stresses for FD	Tractn. in X direct. for BIE	Tractn. in Y direct. for BIE	Tractn. in Z direct. for BIE	equiv. stresses for BIE
A	0.	0.	-0.4269	0.9336	0.	0.	-0.4269	0.9336
B	0.	0.	0.	1.619	0.	0.	0.	1.619
C	0.	0.12606	0.	0.0857	0.	0.12606	0.	0.0857
D	0.	-0.2653	-0.2577	0.2810	0.	-0.2654	-0.2578	0.2810
F	0.19224	-0.0692	0.05376	-0.2515	0.19224	-0.692	0.05376	-0.2515
G	-0.1667	0.17434	0.	0.1657	-0.1667	0.17434	0.	0.1657
H	-0.6790	0.	0.	0.6914	-0.6790	0.	0.	0.6914
I	0.16099	0.	-0.0797	-0.1819	0.16099	0.	-0.0797	-0.1819

Table 1 Traction and equivalent sensitivity results for BIE and FD method for first example.

Nodal points	Displ. in X direction for FD	Displ. in Y direction for FD	Displ. in Z direction for FD	Disp. in X direction for BIE	Disp. in Y direction for BIE	Displ. in Z direction for BIE
A	0.068079	0.55898	0.	0.068079	0.55898	0.
B	5.5621	2.2515	10.065	5.5621	2.2515	10.065
C	0.41694	0.	0.83717	0.41694	0.	0.83717
D	-0.032996	0.	0.	-0.032996	0.	0.
F	0.	0.	0.	0.	0.	0.
G	0.	0.	-0.055929	0.	0.	-0.055929
H	0.	-0.13875	0.48831	0.	-0.13875	0.48831
I	0.	0.047365	0.	0.	0.047365	0.

Table 2 Displacement sensitivity results for BIE and FD method for first example

Nodal points	Tractn. in X direct. for FD E-3	Tractn. in Y direct. for FD E-3	Tractn. in Z direct. for FD E-3	equiv. stresses for FD E-3	Tractn. in X direct. for BIE E-3	Tractn. in Y direct. for BIE E-3	Tractn. in Z direct. for BIE E-3	equiv. stresses for BIE E-3
J	0.	0.	0.1107	0.5114	0.	0.	0.1107	0.5112
K	0.	0.12198	0.	-2.138	0.	0.12187	0.	-2.138
L	1558.92	0.	0.	2638.	1558.92	0.	0.	2638.
M	4.35174	0.	0.	-2.241	4.35172	0.	0.	-2.241
N	0.	0.	0.	94.97	0.	0.	0.	94.97
O	0.	0.	0.	-1.585	0.	0.	0.	-1.585
P	12.0114	0.	-18.705	18.74	12.0114	0.	-18.706	18.74
R	95.3547	-106.71	0.	-7.825	95.3547	-106.71	0.	-7.825

Table 3 Traction and equivalent sensitivity results for BIE and FD method for second example.

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Nodal points	Displ. in X direction for BIE E-3	Displ. in Y direction for BIE E-3	Displ. in Z direction for BIE E-3	Displ. in X direction for FD E-3	Displ. in Y direction for FD E-3	Displ. in Z direction for FD E-3
J	0.62292	-45.217	0.	0.62291	-45.217	0.
K	-32.279	0.	6.5173	-32.279	0.	6.5174
L	0.	1733.3	-2347.9	0.	1733.3	-2347.9
M	0.	-14.445	2.959	0.	-14.445	2.959
N	-35.142	56.731	-71.736	-35.142	56.731	-71.737
O	-14.584	-42.772	14.805	-14.584	-42.772	14.805
P	0.	-96.743	0.	0.	-96.743	0.
R	0.	0.	-87.142	0.	0.	-87.142

Table 4 Displacement sensitivity results for BIE and FD method for second example

5 Conclusions

General BIE formulations for the sensitivity analysis have been developed. Two examples, a cube and a hole in a cube were solved. Traction displacement and equivalent stress sensitivities were obtained. The results obtained are compared between the FDM and BIEM. The BIEM results are shown to be in very close agreement, and for some nodes the results are indistinguishable. Because the general equations are differentiated with respect at the normal of the design variable point, computational time reduced by a factor of three. Comparing the FDM and the BIEM, the latter is significantly faster because it involves only one solution of the equations for each set of sensitivity derivatives, rather than two.

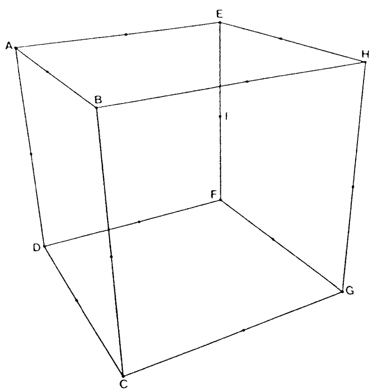

Figure 1 A cube (20 nodes, 6 elements)

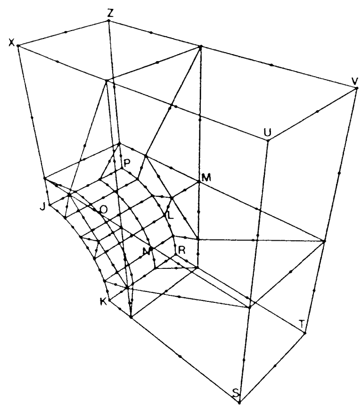

Figure 2 A cube with a cylindrical hole (143 nodes, 51 elements)

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