Dynamic stress intensity factors in mixed-mode: time-domain formulation

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\textbf{Abstract} - The dual boundary element method (DBEM) and the time domain method are applied for the determination of dynamic stress intensity factors (DSIF) for a general mixed-mode crack problem. The DBEM generates a distinct set of boundary integral equations by applying the displacement equation to one of the crack surfaces and the traction equation to the other. The temporal variations of the boundary quantities are either piecewise constant or piecewise linear. The boundary of the body is divided into quadratic elements and quarter-point elements (QPE) are used near the crack tips. The DSIF are calculated using two methods: the crack opening displacements with a least-square error minimization and the path-independent $\bar{J}$-integral with the decomposition technique. The method is applied to the determination of the DSIF of a rectangular plate with an internal inclined crack subjected to impact load. The solution of this mixed-mode crack problem is compared with other reported solution and shows good agreement.

\textbf{Introduction}

The determination of dynamic stress intensity factors (DSIF) plays an important role in fracture mechanics. Peak values of the DSIF are usually much higher than the static values, and accurate methods are needed to calculate them. The application of the boundary element method and the time domain approach to dynamic fracture mechanics has been presented [1 - 6]. Usually symmetric problems are analyzed, where only a half or a quarter of the structure and one of the crack surfaces need to be discretized. For mixed-mode crack problems the structure is divided into subdomains along the crack surfaces and the subdomains are assembled using equilibrium and compatibility conditions. The application of the dual boundary element method in static fracture mechanics was presented by Portela, Aliabadi and Rooke [7]. This method uses two different equations, the displacement and the traction boundary integral equation, for two coincident points on the crack surfaces. The DBEM can solve a general mixed-mode crack

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problem by discretizing the boundary of the structure only. Fedelinski, Aliabadi and Rooke have applied the DBEM combined with the dual reciprocity approach [8, 9], and with a time domain formulation [10] to crack problems in dynamic fracture mechanics.

In the present paper more information about the time domain approach and a new application is presented. The displacement and the traction boundary integral equation are formulated. Information about space and time interpolation and integration are given. As the result of discretization and integration the matrix equation of motion is obtained. The system of equations is solved step-by-step to give the unknown boundary displacements and tractions. Finally an application of the method to a general dynamic mixed-mode crack problem is presented.

**Dual boundary integral equations and time-domain formulation**

Consider a linear elastic homogeneous and isotropic body enclosed by the boundary $\Gamma$. For a body which is not subjected to body forces and which has zero initial displacements and velocities, the displacement of a point $x'$ can be represented by the following boundary integral equation:

$$
\mathbf{c}_{ij}(\mathbf{x}')u_{j}(\mathbf{x}', t) = \int_0^t \left[ \int_{\Gamma} U_{ij}(\mathbf{x}', t; \mathbf{x}, \tau) t_{j}(\mathbf{x}, \tau) d\Gamma(\mathbf{x}) \right] d\tau 
- \int_0^t \left[ \int_{\Gamma} T_{ij}(\mathbf{x}', t; \mathbf{x}, \tau) u_{j}(\mathbf{x}, \tau) d\Gamma(\mathbf{x}) \right] d\tau;
$$

where $U_{ij}(\mathbf{x}', t; \mathbf{x}, \tau)$, $T_{ij}(\mathbf{x}', t; \mathbf{x}, \tau)$ are fundamental solutions of elastodynamics; $u_{j}(\mathbf{x}, \tau)$, $t_{j}(\mathbf{x}, \tau)$ are displacements and tractions respectively, at the boundary; $c_{ij}(\mathbf{x}')$ is a constant which depends on the position of the collocation point $\mathbf{x}'$; $\mathbf{x}$ is the boundary point; $t$ is the observation time.

The fundamental solutions $T_{ij}$ contain Dirac delta functions, as a result of the spatial differentiation of the Heaviside functions in $U_{ij}$. The last term of equation (1), containing $T_{ij}$, is transformed to a form which can be more easily integrated. The spatial derivative of $U_{ij}$ can be expressed by a time derivative, followed by an integration by parts [6]. After this modification the new form of the displacement boundary integral equation is

$$
\mathbf{c}_{ij}(\mathbf{x}')u_{j}(\mathbf{x}', t) = \int_0^t \left[ \int_{\Gamma} U_{ij}(\mathbf{x}', t; \mathbf{x}, \tau) t_{j}(\mathbf{x}, \tau) d\Gamma(\mathbf{x}) \right] d\tau 
- \int_0^t \left[ \int_{\Gamma} \left( T_{ij}(\mathbf{x}', t; \mathbf{x}, \tau) u_{j}(\mathbf{x}, \tau) - \hat{T}_{ij}(\mathbf{x}', t; \mathbf{x}, \tau) \mathbf{u}_{j}(\mathbf{x}, \tau) \right) d\Gamma(\mathbf{x}) \right] d\tau,
$$

where $T_{ij}$ and $\hat{T}_{ij}$ are expressions given in [6]; and a dot above a variable denotes the derivative with respect to time.

The traction equation is obtained by differentiating the displacement equation, applying Hooke's law and multiplying by the normal at the collocation point. For a point which belongs to the smooth boundary the traction equation is
Boundary Element Method XVI

\[ \frac{1}{2} t_j(x', t) = n_i(x') \left\{ \int_0^t \left[ \int_{\Gamma} U_{kij}^m(x', t; x, \tau)t_j^m(x, \tau) d\Gamma(x) \right] d\tau \right. \]

\[ \left. - \int_0^t \left[ \int_{\Gamma} T_{kij}^m(x', t; x, \tau)u_k(x, \tau) d\Gamma(x) \right] d\tau \right\}, \]

where \( n_i(x') \) are components of the outward normal at the collocation point \( x' \) and \( U_{kij}^m(x', t; x, \tau) \), \( T_{kij}^m(x', t; x, \tau) \) are other fundamental solutions of elastodynamics.

Numerical formulation

The numerical solution of a general mixed-mode crack problem is obtained after discretizing both space and time variations. The boundary \( \Gamma \) of the body is divided into \( M \) boundary elements with \( P \) nodes per element. The observation time \( t \) is divided into \( N \) time steps. The temporal variation of boundary quantities is specified by \( Q \) values within the time step. Displacements and tractions are approximated within each element using interpolation functions \( N^p(\xi) \) and within each time step using interpolation functions \( M^q(\tau) \). The boundary integral equations are applied for all the nodes of the boundary elements. After the approximation, the displacement and the traction equation are:

\[ c_{ij}^l u_j^{ln} = \sum_{m=1}^M \sum_{p=1}^P \sum_{n=1}^N \sum_{q=1}^Q \left\{ t_j^{mpq} \int_{t_n-1}^{t_n} \left[ \int_{t_{n-1}}^{t_n} U^N_{ij}(\xi, \tau) M^q(\tau) d\tau \right] N^p(\xi) J^m(\xi) d\xi \right. \]

\[ \left. - u_j^{mpq} \int_{t_{n-1}}^{t_n} \left[ \int_{t_{n-1}}^{t_n} \left( \check{T}_{ij}^N(\xi, \tau) M^q(\tau) - \check{T}_{ij}^N(\xi, \tau) \check{M}^q(\tau) \right) d\tau \right] N^p(\xi) J^m(\xi) d\xi \right\}, \quad l = 1, 2, ..., L_1 \]

\[ \frac{1}{2} \tilde{t}_j^l = n_i \sum_{m=1}^M \sum_{p=1}^P \sum_{n=1}^N \sum_{q=1}^Q \left\{ t_k^{mpq} \int_{t_n-1}^{t_n} \left[ \int_{t_{n-1}}^{t_n} U^N_{kij}(\xi, \tau) M^q(\tau) d\tau \right] N^p(\xi) J^m(\xi) d\xi \right. \]

\[ \left. - u_k^{mpq} \int_{t_{n-1}}^{t_n} \left[ \int_{t_{n-1}}^{t_n} T_{kij}^N(\xi, \tau) M^q(\tau) d\tau \right] N^p(\xi) J^m(\xi) d\xi \right\}, \quad l = 1, 2, ..., L_2. \]

where \( L_1 \) and \( L_2 \) are the numbers of collocation points for which the displacement and the traction equations are applied, respectively, and \( L_1 + L_2 = L \) is the total number of nodes; \( J^m \) is the Jacobian and \( \xi \) is the local coordinate.

The distinct set of the boundary integral equations is obtained by applying the displacement equation (4) for the collocation points along the external boundary \( \Gamma_a \) and along one of the crack faces \( \Gamma_b \), and the traction equation (5) for the opposite surface of the crack \( \Gamma_c \) (see Fig. 1).

Quadratic elements are used for the discretization of the boundary. The displacements and tractions are interpolated using: continuous elements for the
external boundary $\Gamma_a$, semi-discontinuous elements at junctions with the cracks, straight discontinuous elements on the crack faces $\Gamma_b$ and $\Gamma_c$. The geometry is approximated by using continuous elements.

The shape functions for the continuous elements ($\xi' = -1, 0, 1$) are:

$$N^1(\xi) = \frac{1}{2} \xi(\xi - 1), \quad N^2(\xi) = 1 - \xi^2, \quad N^3(\xi) = \frac{1}{2} \xi(\xi + 1),$$

(6)

for the semi-discontinuous elements ($\xi' = -\frac{2}{3}, 0, 1$) are:

$$N^1(\xi) = \frac{9}{10} \xi(\xi - 1), \quad N^2(\xi) = -\frac{3}{2} \xi^2 + \frac{1}{2} \xi + 1, \quad N^3(\xi) = \frac{1}{5} \xi(3\xi + 2),$$

(7)

and for the discontinuous elements ($\xi' = -\frac{2}{3}, 0, \frac{2}{3}$) are:

$$N^1(\xi) = \frac{3}{4} \xi(\frac{3}{2} \xi - 1), \quad N^2(\xi) = 1 - \frac{9}{4} \xi^2, \quad N^3(\xi) = \frac{3}{4} \xi(\frac{3}{2} \xi + 1),$$

(8)

where $\xi'$ is the local coordinate of the node.

At the crack tips quarter-point elements (QPE) are used. For the QPE the original mid-node of the element approximating the geometry is moved to a quarter of the length of the element. The local coordinate of the distorted element is a square-root function of the distance $r$ from the crack tip

$$\xi = 1 - 2 \sqrt{r \frac{l}{l}},$$

(9)

where $l$ is the length of the element and the crack tip is at $\xi = 1$. The QPE better represents the square-root behaviour of the displacements near the crack tips.

For the displacement equation (3) the displacements are approximated within each time step by using linear interpolating functions and the tractions are piecewise constant. The mixed variation used for the displacement equation gives a better solution when the structure is subjected to impact loads [6]. For the traction equation (4), both the displacements and the tractions are assumed to be piecewise linear.

The constant temporal shape function is

$$M^1(\tau) = 1$$

(10)

and the linear temporal shape functions are:

$$M^1(\tau) = \frac{\tau - t_{n-1}}{\Delta t}, \quad M^2(\tau) = \frac{t_n - \tau}{\Delta t},$$

(11)

where $t_{n-1} \leq \tau \leq t_n, \Delta t$ is the time step and the superscripts denote the forward and the backward local time node, respectively. The time integrals are calculated analytically as shown for the displacement equation in [6] and for the stress equation in [4]. The convoluted fundamental solutions have spatial singularities during the first time step and their orders are the same as the static fundamental solutions. The singularities are of $O(ln(r))$ and $O(1/r)$ in the displacement equation and $O(1/r)$ and $O(1/r^2)$ in the traction equation, where $r$ is the distance.
from the collocation point. The coefficients $c_{ij}$ are calculated analytically \[6\]. The boundary integrals are integrated semi-analytically or analytically when the collocation point belongs to the element and numerically for other elements using Gaussian integration.

After the discretization and integration the following matrix equation is obtained

$$\sum_{n=1}^{N-1} (G^{Nn}t^n - H^{Nn}u^n),$$

where $u^n$, $t^n$ contain nodal values of displacements and tractions at the time step $n$, $H^{Nn}$, $G^{Nn}$ depend on fundamental solutions and interpolating functions. The superscripts $Nn$ emphasize that the matrix depends on the difference between the time step $N$ and $n$. The columns of matrices $H^{Nn}$, $G^{Nn}$ are reordered according to the boundary conditions, giving new matrices $A^{NN}$ and $B^{NN}$. The matrix $A^{NN}$ is multiplied by the vector $X^N$ of unknown displacements and tractions and the matrix $B^{NN}$ by the vector $Y^N$ of known boundary conditions, as follows

$$A^{NN}X^N = B^{NN}Y^N + \sum_{n=1}^{N-1} (G^{Nn}t^n - H^{Nn}u^n).$$

In each time step only the matrices, which correspond to the maximum difference $N-n$ are computed. The rest of the matrices is known from the previous steps. The matrices $A^{NN}$ and $B^{NN}$ are calculated in the first step only since they are the same at each time step; $A^{NN} = A$ and $B^{NN} = B$. The matrix equation (13) can be written in a simpler form as

$$AX^N = F^N,$$

where

$$F^N = BY^N + \sum_{n=1}^{N-1} (G^{Nn}t^n - H^{Nn}u^n)$$

is the known vector. The matrix equation is solved step-by-step giving the unknown displacements and tractions at each time step. During the initial steps the fundamental solutions are non-zero only in the neighbourhood of the collocation point; they are therefore integrated only over that part of the boundary. The solution process becomes slower at later times because the vector $F^N$ depends on all the matrices from the previous steps. More information about the numerical implementation of the method will be presented in [11].

**Numerical example**

In order to demonstrate the applicability of the method the following mixed-mode crack problem was considered.

A rectangular plate of length $2b = 60 mm$ and width $2h = 30 mm$ contains a central inclined crack of length $2a = 14.14 mm$ slanted at an angle $\alpha = 45^\circ$, as shown in Fig. 2. The plate is instantaneously loaded by a uniform tensile stress $\sigma_0$ at time $t = 0$. The plate has the following material properties: the shear modulus
$\mu = 76.92 \cdot 10^9 \, Pa$; Poisson’s ratio $\nu = 0.3$; and the density $\rho = 5000 \, kgm^{-3}$. A state of plane strain is assumed. The boundary is divided into 40 boundary elements and the time step $\Delta t = 0.8 \, \mu s$.

![Figure 1: Modelling of the boundary](image1)

![Figure 2: Rectangular plate with an internal inclined crack](image2)

The dynamic stress intensity factors are calculated using two methods. The first method minimizes the sum of the squared differences between the analytical and numerical values of the crack opening displacements of the nodes near the crack tip [10]. The second method determines the path-independent $J$-integral from the decomposition technique and uses a regular polygonal path with the centre at the crack tip. The internal values of the spatial derivatives of displacements, stresses and accelerations, required for the $J$-integral, are obtained by using time-domain boundary integral equations. The numerical implementation of the path-independent integral is similar to that applied by the authors [9] for the DBEM and the dual reciprocity method. The DSIF are normalized with respect to

$$K_o = \sigma_o \sqrt{\pi a}.$$ (16)

The results are presented in Figs 3 and 4 and compared with those of Dominguez and Gallego [5], who used the time domain formulation of the BEM and the subregion technique. The DSIF obtained by the COD and the $J$-integral are similar and agree well with those results.

Conclusions

The dual boundary element method and the time domain approach are applied to determine the dynamic stress intensity factors. In this formulation, only the boundary of the structure needs to be discretized. The DSIF are calculated using the crack opening displacements and a path-independent integral. The method has been applied to a dynamic mixed-mode crack problem and the solution compared with an other solution obtained by the subregion method. Both solutions agree well showing that the method can be used efficiently.
Figure 4: Normalized dynamic stress intensity factors $K_{II}/K_o$ for the rectangular plate with an internal inclined crack.

Figure 3: Normalized dynamic stress intensity factors $K_{I}/K_o$ for the rectangular plate with an internal inclined crack.
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References


