A hybrid displacement variational formulation of BEM for elastodynamics

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Abstract

This paper deals with a hybrid displacement variational formulation of BEM for elastodynamics. The mathematical model for elastodynamics is obtained from the stationarity condition of a modified functional expressed in terms of displacements and tractions. These are assumed as independent on the boundary and in the domain. The boundary variables are expressed by using their nodal values while the domain displacement field is approximated with a linear combination of static fundamental solutions. The source point of every fundamental solution is located outside the domain. The problem of free vibrations is reduced to an algebraic eigenvalue problem, the solution of which is straightforward. The structural matrices result symmetric and positive definite and the domain integral, associated to the inertial term, can be transformed into a boundary one. Numerical results are in excellent agreement with existing ones.

1 Introduction

The boundary element method BEM has a wide field of applications and it was successfully applied to many structural problems. BEM, in its own direct approach, is based on the solution of an integral equation system. This system is obtained by using the so called 'fundamental solutions' that are solutions of the structural problem when a singular action is imposed on the considered body. The direct approach of BEM leads to a resolving system in which, generally, certain tensor operators lose some fundamental properties of continuum: symmetry and definiteness are violated. Moreover in the direct formulation of BEM for elastodynamics the resolving system presents a domain integral associated to the inertial term. The last problem can be avoided by using transcendental fundamental solutions that are implicit functions of vibration frequencies. However the use of this kind of fundamental solution
leads to a non-linear system that can be solved only by means of very approximate iterative methods [1-2]. Nardini and Brebbia [3-4] have proposed an approach where the fundamental solutions are frequency independent. Such a condition, with an appropriate choice of auxiliary functions, allows the transformation of the domain integral into a boundary one and yields a linear resolving system. Anyway, as direct BEM and D/BEM [5-6] destroy symmetry and definiteness of some tensor operators, their applicability to dynamic and non-linear problems is limited. For this reason some authors [7-10] have concentrated their surveys on the development of variational formulations.

In this paper a symmetric and positive definite BEM formulation, proposed by Davi [10], is developed for elastodynamics. In this formulation the dynamic mathematical model is obtained by using a modified functional in which boundary and domain variables are taken independent of each another. The boundary variables are expressed by their nodal values whereas the domain ones are approximated with a linear combination of regular static fundamental solutions. The resolving system results in an algebraic one. The stiffness and mass matrices are symmetric and positive definite. The domain integral, associated to the inertial term, is transformed into a boundary integral and all the structural matrices can be computed performing only boundary integrations.

### 2 Governing equations for Elastodynamics

Consider an isotropic, homogeneous, linear elastic body subjected in the domain $\Omega$ to general time dependent body forces $f$. The body is constrained on the boundary $\Gamma_1$ and let it be loaded by traction $t$ on the free boundary $\Gamma_2$. Let $u$, $\varepsilon$, $\sigma$ be the displacement, strain and stress fields in $\Omega$. The mathematical model of the dynamic response of the body is made by the following equations:

**Equilibrium**

$$D^T\sigma + f - \mu \ddot{u} - \rho \ddot{u} = 0 \quad \text{in } \Omega \tag{1}$$

$$t = D_n \sigma \quad \text{on } \Gamma \tag{2}$$

**Compatibility**

$$\varepsilon = Du \quad \text{in } \Omega \tag{3}$$

**Constitutive**

$$\sigma = E\varepsilon \quad \text{in } \Omega \tag{4}$$

where

$$D^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \tag{5}$$

$$D_n = \begin{bmatrix} \alpha_1 & 0 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_2 & 0 & \alpha_1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_3 & 0 & \alpha_1 & \alpha_2 \end{bmatrix} \tag{6}$$

In the previous expressions $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the direction cosines of the outward normal on the boundary, $\rho$ is the mass density and $\mu$ is the damping
Boundary Element Method XVI

Coefficient. The mathematical model is completed by the boundary and the initial conditions that are

\[ u = \tilde{u} \quad \text{on } \Gamma_1 \]  
\[ t = \tilde{t} \quad \text{on } \Gamma_2 \]  
\[ \tilde{u}(\tau = 0) = \tilde{u}_0 \quad ; \quad u(\tau = 0) = u_0 \]  

3 Modified variational principle

Let \( u \) be the displacement field in the domain \( \Omega \) and \( \tilde{u} \) and \( \tilde{t} \) the displacements and the tractions on the boundary \( \Gamma \) (\( \Gamma = \Gamma_1 + \Gamma_2 \)). It is assumed that the functions \( u \), \( \tilde{u} \) and \( \tilde{t} \) are independent from one another. The used modified variational principle states that the solution of the dynamic problem makes the following functional stationary

\[ \Pi = \int_{\Omega} \left( \frac{1}{2} \varepsilon^T \varepsilon - u^T f' \right) \, d\Omega - \int_{\Gamma_1} (u - \tilde{u})^T \tilde{t} \, d\Gamma - \int_{\Gamma_2} \tilde{u}^T \tilde{t} \, d\Gamma \]  

where

\[ f' = f(\tau) - \mu \tilde{u} - \rho \tilde{u} \]  

In eqn (10) \( \tilde{t} \) are the tractions prescribed on the free boundary \( \Gamma_2 \), while on the constrained boundary \( \Gamma_1 \) the condition is

\[ \tilde{u} = \tilde{u} \quad \text{on } \Gamma_1 \]  

Taking the variation with respect to the independent variables one obtains:

\[ \partial \Pi = -\int_{\Omega} \left( D^T \sigma + f' \right) \partial u \, d\Omega - \int_{\Gamma_1} (u - \tilde{u})^T \partial \tilde{t} \, d\Gamma + \int_{\Gamma_2} \tilde{t}^T \partial \tilde{u} \, d\Gamma \]  

and the stationarity condition yields:

\[ D^T \sigma + f' = 0 \quad \text{in } \Omega \]  
\[ \tilde{u} = \tilde{u} \quad \text{on } \Gamma_1 \]  
\[ \tilde{t} = \tilde{t} \quad \text{on } \Gamma_1 \]  
\[ \tilde{t} = \tilde{t} \quad \text{on } \Gamma_2 \]

Thus assuming that eqns (2), (3), (4) are verified and that eqn (9) and eqn (12) are identically satisfied, the solution to the dynamic problem is given in terms of the functions \( u \), \( \tilde{u} \) and \( \tilde{t} \) that make \( \Pi \) stationary.

4 BEM formulation

Let us suppose that the body boundary \( \Gamma \) is divided into a finite number of elements. Because of such discretization the displacement and traction functions on the boundary can be expressed by means of their nodal values:
In the previous relationships $\mathbf{N}$ and $\Psi$ are matrices of shape functions while $\mathbf{\delta}$ and $\mathbf{p}$ are the nodal displacements and nodal tractions, respectively. The displacement field in the domain $\Omega$ is built up by a linear combination of static fundamental solutions, the source point of which is set outside the domain. It can be written

$$u(\tau) = \mathbf{u}^* \mathbf{s}(\tau) \quad (20)$$

where $\mathbf{s}$ is a vector of time dependent coefficient and $\mathbf{u}^*$ is the matrix of the fundamental solutions. Partitioning $\mathbf{\delta}$ we have:

$$\mathbf{\delta} = \begin{bmatrix} \mathbf{\delta}_1 \\ \mathbf{\delta}_2 \end{bmatrix} \quad (21)$$

and consequently the condition on the constrained boundary becomes

$$\mathbf{\delta}_1 = \mathbf{\bar{\delta}}_1 \quad \text{on } \Gamma_1 \quad (22)$$

The modified functional, with these assumptions, takes the following form:

$$\Pi = \frac{1}{2} \mathbf{s}^T \int_{r} \mathbf{u}^* \mathbf{p}^* \mathbf{d}\Gamma - \mathbf{p}^T \int_{r} \Psi^T \mathbf{u}^* \mathbf{d}\Gamma \mathbf{s} + \mathbf{p}^T \int_{r} \Psi^T \mathbf{N} \mathbf{d}\Gamma \mathbf{\delta} + \int_{r_2} \mathbf{N}^T \mathbf{t} \mathbf{d}\Gamma - \mathbf{s}^T \int_{\Omega} \mathbf{u}^* \mathbf{f}^* \mathbf{d}\Omega \quad (23)$$

where

$$\mathbf{p}^* = \mathbf{D}_Q \mathbf{E} \mathbf{u}^* \quad (24)$$

Stationarity of $\Pi$ with regard to $\mathbf{s}, \mathbf{\delta}$, and $\mathbf{p}$ yields

$$\int_{r} \mathbf{u}^* \mathbf{p}^* \mathbf{d}\Gamma \mathbf{s} - \int_{r} \mathbf{u}^* \Psi^T \mathbf{p} \mathbf{d}\Gamma \mathbf{s} + \int_{\Omega} \mathbf{u}^* \mathbf{f} \mathbf{d}\Omega = 0 \quad (25)$$

$$\int_{r} \mathbf{N}^T \Psi \mathbf{d}\Gamma \mathbf{p} - \int_{r_2} \mathbf{N}^T \mathbf{t} \mathbf{d}\Gamma = 0 \quad (26)$$

$$\int_{r} \Psi^T \mathbf{u}^* \mathbf{d}\Gamma \mathbf{s} - \int_{r} \Psi^T \mathbf{N} \mathbf{d}\Gamma \mathbf{\delta} = 0 \quad (27)$$

Eqn (27) is satisfied for every choice of $\Psi$ if

$$\mathbf{u}^* \mathbf{s} = \mathbf{N} \mathbf{\delta} \quad (28)$$

then, computing eqn (28) for the nodal points, one obtains:

$$\mathbf{\bar{u}}^* \mathbf{s} = \mathbf{\delta}$$

and, because $\mathbf{\bar{u}}^*$ is non singular, the previous relationships give us:

$$\mathbf{s} = (\mathbf{\bar{u}}^*)^{-1} \mathbf{\delta} = \Phi \mathbf{\delta} = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \quad (30)$$
By pre-multiplying eqn (25) by \( \Phi^T \) and taking into account eqn (30) and eqn (26) the following resolving system is achieved:

\[
K \delta_i = \int_{\Omega_2} N_i^T \Gamma d\Gamma + \int_{\Omega_2} N_i^T \Gamma' d\Omega - \Phi^T A \Phi \delta_i
\]  

(32)

where

\[
K = \Phi^T \int_{\Gamma} u^T p^* d\Gamma \Phi
\]  

(33)

System (32) is the discrete model for elastodynamics in which the stiffness matrix \( K \) and the mass matrix, defined as

\[
M = \rho \int_{\Omega} N_i^T N_i d\Omega
\]  

(34)

are symmetric and positive definite. For the free vibration analysis eqn (32) becomes:

\[
M \ddot{\delta}_i + K \delta_i = 0
\]  

(35)

The last expression represents a classic algebraic eigenvalue problem that, since both the matrices \( K \) and \( M \) are symmetric and positive definite, can be solved by means of simple numeric standard routines.

5 Transformation of the domain integral into a boundary integral

The evaluation of mass matrix is carried out by converting the domain integral of eqn (34) into a boundary one. This goal can be obtained by using the reciprocity theorem. Let us consider a system of body forces \( b = u^* \); let \( v \) be a generic field of displacements associated to \( b \) and let \( q \) be the corresponding vector of tractions. Carrying out the reciprocity theorem one obtains:

\[
\int_{\Omega} [u^T b - f^* v] d\Omega + \int_{\Gamma} [u^T q - p^* v] d\Gamma = 0
\]  

(36)

As \( f^* = 0 \) and \( b = u^* \) in \( \Omega \), eqn (36) becomes:

\[
\int_{\Omega} u^T u d\Omega = \int_{\Gamma} [p^* v - u^T q] d\Gamma
\]  

(37)

Hence, generalizing for all fundamental solutions, the mass matrix can be expressed as

\[
M = \rho \Phi^T B \Phi
\]  

(38)

where

\[
B = \int_{\Gamma} [p^* v - u^T q] d\Gamma
\]  

(39)
Therefore the problem of transforming the domain integral into a boundary one can be solved finding a displacement field associated to the body forces $b = u^*$ for every fundamental solution $u^*$.

6 Numerical applications and results

To show the accuracy of the proposed method, two in-plane free vibration applications are presented. The obtained results are compared with those obtained out of FEM. For all the examples the boundary is discretized using linear elements. The nodal point is the mid-point of the element and the fundamental solution source point is located outside the domain, along the outer normal at the nodal point. The distance of the source point from the element can be chosen arbitrarily because the final result is independent from this parameter. The plain strain fundamental solution is employed. All structural matrices are calculated only by boundary integration, using the Gauss quadrature formulas. The free vibration analysis of a cantilever beam was carried out. In fig. 1 are shown the convergence curves of periods when the number of elements increases. In this case the first five modes are taken into account. In the second example a shear wall, showed in fig. 2, is considered. The shear wall vibration periods are reported in table 1 and are compared with the ones obtained by means of FEM, with a mesh of 599 nodes. The BEM analysis is carried out with only 73 nodes. The diagrams prove the excellent agreement of the present result with the existing ones also when very few elements are employed. However, notice that the proposed method, with its approximations and characteristics, will be sufficient for determining modes of vibration with low frequencies.

7 Conclusion

A new approach of BEM for elastodynamics has been presented in this paper. The variational formulation is expressed in terms of domain and boundary variables, considered independent from one another. The final system involves boundary displacement only and both stiffness and mass matrices are symmetric and positive definite. These matrices are frequency independent and then the free vibration analysis is reduced to an algebraic eigenvalue problem. Moreover the approach effectively permits the mass matrix, which is originally obtained as a domain integral, to be calculated as a boundary integral. The numerical results carried out for the free vibration case show the accuracy and the efficiency of the proposed formulation. The convergence rate is dependent on the ratio between free and restrained boundary nodes: when the boundary is not restrained very much the results are accurate also for a very low number of elements. Although it is pointed out that the method will be sufficient for determining modes of vibration with low frequencies that, however, are the most important from a structural standpoint. The proposed formulation represents a suitable means of investigation for dynamic problems among which there are transient cases and a wide variety of time dependent problems.
Figure 1: Period convergence curves for a cantilever beam.

Figure 2: Shear wall geometry and BEM discretization.

Table 1. Periods of free vibration for the shear wall.

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>FEM 599 nodes</td>
<td>3.029</td>
<td>0.885</td>
<td>0.824</td>
<td>0.526</td>
</tr>
<tr>
<td>BEM 73 nodes</td>
<td>3.050</td>
<td>0.913</td>
<td>0.811</td>
<td>0.480</td>
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References


