A fictitious stress and displacement discontinuity method for dynamic crack problems

P.H. Wen\(^{a,*}\), M.H. Aliabadi\(^{a}\) & D.P. Rooke\(^{b}\)

\(^{a}\)Wessex Institute of Technology, Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK
\(^{b}\)DRA, Farnborough, Hants, GU14 6TD, UK

Abstract

In this paper, the fictitious stress and displacement discontinuity integral equations are presented in the Laplace transform domain. The two formulations are used to generate an indirect boundary integral equation method for the solution of elastodynamic crack problems. The proposed method is used to evaluate the mixed mode stress intensity factors for several crack configurations subjected to dynamic loading.

1 Introduction

The boundary integral equation methods have in recent years proved very successful in solving static crack problems in fracture mechanics; see Aliabadi and Rooke\(^{[1]}\) for a review. However, their application to dynamic crack problems is relatively new. The so-called direct boundary integral formulations have been applied to crack problems by many researchers, for example Dominguez and Gallego\(^{[2]}\) using the multi-region formulation and Fedelinski, Aliabadi and Rooke\(^{[3]}\) using a single region formulation. An alternative approach to the direct formulation is the indirect formulation which relates the physical parameters, such as displacements and tractions, to fictitious source densities. Two indirect formulations known as the fictitious stress and the displacement discontinuity methods were developed by Crouch and Starfield\(^{[4]}\) to study fracture of rocks in geomechanical problems. In this paper, these two formulations have been extended to elastodynamic crack problems.

2 Fictitious Stress Method

The solution to a general problem shown in figure 1a, can be obtained in an indirect way by considering an infinite plane containing the boundary \(\Gamma\) with distributed loads \(p(x, \tau_k)\) and \(q(x, \tau_k)\) along the normal and tangential directions respectively (see figure 1b); where \(x\) denotes the spatial coordinates and \(\tau_k\) the discrete Laplace transform parameter.

The stresses \(\sigma_{ij}\) and displacements \(u_i\) due to the fictitious loads can be obtained from the solution of concentrated forces \(\sigma_{ij}^*\) and \(u_i^*\) for \(F_x\) and \(\sigma_{ij}^{**}\) and \(u_i^{**}\) for \(F_y\) (see ref\(^{[9]}\)). Let \(F_x = qdy\) and \(F_y = pdx\) with \(p\) and \(q\) uniformly distributed over a line of length \(2a\); the solution for stresses and displacements can be obtained from

\[
\sigma_{ij} = \frac{p}{F_y} \int_{-a}^{a} \sigma_{ij}^*(x-\eta, y, \tau) d\eta + \frac{q}{F_x} \int_{-a}^{a} \sigma_{ij}^{**}(x-\eta, y, \tau) d\eta
\]

and

\(*on leave from Central-South University of Technology, Chang Sha, China."
Figure 1: a) Two-dimensional geometry with arbitrary boundary conditions, b) fictitious loads

\[ u_i = \frac{p}{F_y} \int_{-a}^{a} u_i^*(x - \xi, y, \tau)d\eta + \frac{q}{F_x} \int_{-a}^{a} u_i^{**}(x - \eta, y, \tau)d\eta \]  

Substitution of the concentrated load solutions (see Ref[9]), gives

\[ \sigma_{xx} = \frac{p}{2\pi \beta_2^2} \left[ \frac{\partial^2 k}{\partial x \partial y} - \frac{\partial^2 \tilde{k}}{\partial x \partial y} \right]_{r-a}^{r+a} + \frac{q}{2\pi \beta_2^2} \left[ \lambda \beta_1^2 k + 2\mu \left( \frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 \tilde{k}}{\partial y^2} \right) \right]_{r-a}^{r+a} \]

\[ \sigma_{yy} = -\frac{p}{2\pi \beta_2^2} \left[ \frac{\partial^2 k}{\partial x \partial y} - \frac{\partial^2 \tilde{k}}{\partial x \partial y} \right]_{r-a}^{r+a} + \frac{q}{2\pi} \int_{r-a}^{r+a} \frac{\partial k}{\partial y} d\eta \]

\[ \sigma_{xy} = \frac{p}{2\pi} \left( \frac{1}{\beta_2^2} \right) \left[ 2 \left( \frac{\partial^2 k}{\partial y^2} - \frac{\partial^2 \tilde{k}}{\partial y^2} \right) \right]_{r-a}^{r+a} + \frac{q}{2\pi} \int_{r-a}^{r+a} \frac{\partial k}{\partial y} d\eta \]

\[ u_x = \frac{p}{2\pi \beta_2^2} \left[ \frac{\partial k}{\partial y} - \frac{\partial \tilde{k}}{\partial y} \right]_{r-a}^{r+a} + \frac{q}{2\pi} \int_{r-a}^{r+a} \frac{\partial k}{\partial y} d\eta + \frac{q}{2\pi \beta_2^2} \left[ \frac{\partial k}{\partial x} - \frac{\partial \tilde{k}}{\partial x} \right]_{r-a}^{r+a} \]
for displacements; where \( \beta_2 = \frac{\tau}{c_2}, k = k_0(\beta_1 r), \tilde{k} = k_0(\beta_2 r), k_0 \) is the zero-order Bessel's function, \( \beta_1 = \frac{\tau}{c_1} \) and \( r = \sqrt{(x - \eta)^2 + y^2}. \) The propagation velocities of dilatation and distortional wave are denoted as \( c_1 \) and \( c_2; \mu \) and \( \lambda \) are the Lamé constants, with \( \mu \) the shear modulus.

Now assuming that the boundary \( \Gamma \) is divided into \( N_T \) flat segments and fictitious loads \( p \) and \( q \) are taken to be constant over them, the stresses at a point \( x^i \) on \( \Gamma \) can be written as

\[
\sigma_n^i = \sum_{j=1}^{N_T} A_{nn}^{ij} p^j + \sum_{j=1}^{N_T} A_{ns}^{ij} q^j
\]

\[
\sigma_s^i = \sum_{j=1}^{N_T} A_{sn}^{ij} p^j + \sum_{j=1}^{N_T} A_{ss}^{ij} q^j;
\]

where \( i = 1, 2, ..., N_T \) and subscripts \( n, s \) denote the normal and tangential directions in the local coordinate system for each element. \( A_{nn}^{ij}, A_{ns}^{ij}, A_{sn}^{ij} \) and \( A_{ss}^{ij} \) are influence functions. For example \( A_{nn}^{ij} \) represents the normal stress on element \( i \) when there is a uniform unit normal load (i.e. \( p^j = 1, q^j = 0 \)) on element \( j \). The terms \( p^j \) and \( q^j \) represent the fictitious loads at the mid-point of each element. The influence functions are obtained from equation (3) through the usual global-local coordinate transformations. It is worth noting that when \( i = j, A_{nn}^{ii} = A_{ss}^{ii} = -0.5 \) and \( A_{sn}^{ii} = A_{ns}^{ii} = 0. \)

The displacements \( u \) at a point \( x^i \) on the boundary \( \Gamma \) can be written in a similar way as

\[
u_n^i = \sum_{j=1}^{N_T} U_{nn}^{ij} p^j + \sum_{j=1}^{N_T} U_{ns}^{ij} q^j
\]

\[
u_s^i = \sum_{j=1}^{N_T} U_{sn}^{ij} p^j + \sum_{j=1}^{N_T} U_{ss}^{ij} q^j,
\]

where \( i = 1, 2, ..., N_T; U_{nn}^{ij}, U_{ns}^{ij}, U_{sn}^{ij} \) and \( U_{ss}^{ij} \) are the displacement influence functions and can be obtained from (4).

### 3 Displacement Discontinuity Method

The displacement discontinuity method utilizes the fundamental solutions due to a constant discontinuity in the displacements over a finite line segment of a crack surface with a condition that the displacements be continuous everywhere else. The displacement discontinuity \( (u_I, u_{II}) \) are defined as the difference in the displacement between the crack surfaces, that is

\[
u_I = \nu_n^+ - \nu_n^- \quad -a \leq x \leq a
\]

\[
u_{II} = \nu_s^+ - \nu_s^- \quad -a \leq x \leq a
\]

where \( \pm \) refers to the upper and lower crack surfaces respectively.

In a similar way to the fictitious stress method, the stresses and displacements due to \( (u_I, u_{II}) \) can be obtained (see ref[9]). They are given as
\[
\sigma_{xx} = \frac{2u_I}{\pi} \left[ \left( -\mu + \lambda \left( \frac{\beta_1}{\beta_2} \right)^2 \right) \frac{\partial k}{\partial x} + \frac{2\mu}{\beta_2^2} \left( \frac{\partial^3 k}{\partial x^3} + \frac{\partial^3 \tilde{k}}{\partial x \partial y^2} \right) \right]_{x=a}^{x+a} - \frac{\lambda \beta_1^2 u_I}{\pi} \int_{x=a}^{x+a} k \, d\eta
\]

\[
\sigma_{yy} = \frac{2u_I}{\pi} \left[ \left( \mu + \lambda \left( \frac{\beta_1}{\beta_2} \right)^2 \right) \frac{\partial k}{\partial x} + \frac{2\mu}{\beta_2^2} \left( \frac{\partial^3 k}{\partial x^2 \partial y} - \frac{\partial^3 \tilde{k}}{\partial x \partial y^2} \right) \right]_{x=a}^{x+a}
\]

\[
\sigma_{xy} = \frac{2u_I}{\pi} \left[ \frac{\partial^3 \tilde{k}}{\partial x \partial y^2} - \frac{\partial^3 k}{\partial x^2 \partial y} - \frac{\partial^3 \tilde{k}}{\partial x^2 \partial y} + \frac{\partial^3 \tilde{k}}{\partial x \partial y^3} \right]_{x=a}^{x+a} - \frac{\mu \beta_2^2 u_{II}}{\pi} \int_{x=a}^{x+a} k \, d\eta
\]

As in the fictitious stress method the boundary is divided into straight segments, say \(N_c\).

The stress and displacement components along the normal and tangential directions can be written as follows:

\[
\sigma_n^i = \sum_{j=1}^{N_c} D_{nn}^{ij} u_{n}^i + \sum_{j=1}^{N_c} D_{nn}^{ij} u_{nII}^i,
\]

\[
\sigma_s^i = \sum_{j=1}^{N_c} D_{ns}^{ij} u_{n}^i + \sum_{j=1}^{N_c} D_{ns}^{ij} u_{nII}^i;
\] (10)
where $D_{ij}^{nn}$, $D_{ij}^{ns}$, $D_{ij}^{ss}$, and $D_{ij}^{nj}$ are the stress influence functions; $V_{ij}^{nn}$, $V_{ij}^{ns}$, $V_{ij}^{ss}$, and $V_{ij}^{nj}$ are the displacement influence functions. These influence functions can be computed from equations (8) and (9).

4 Mixed Formulation for Cracked Bodies in Finite Bodies

The application of the fictitious stress formulation alone to crack problems results in a singular system due to the coincident crack surfaces. In order to avoid this difficulty, the displacement discontinuity solutions are used on the crack surfaces $\Gamma_c$ and the fictitious stress solutions on the remaining boundary $\Gamma$, to give

$$\sigma_i^j = \sum_{j=1}^{N_T} A_{ij}^{nn} p^j + \sum_{j=1}^{N_T} A_{ij}^{ns} q^j + \sum_{j=1}^{N_T} D_{ij}^{nn} u_i^j + \sum_{j=1}^{N_T} D_{ij}^{ns} u_{II}^j$$

and

$$\sigma_s^i = \sum_{i=1}^{N_T} A_{ii}^{ns} p^i + \sum_{i=1}^{N_T} A_{ii}^{ss} q^i + \sum_{i=1}^{N_T} D_{ii}^{nn} u_i^i + \sum_{i=1}^{N_T} D_{ii}^{ss} u_{II}^i$$

for the stresses; and

$$u_i^j = \sum_{j=1}^{N_T} U_{ij}^{nn} p^j + \sum_{j=1}^{N_T} U_{ij}^{ns} q^j + \sum_{j=1}^{N_T} V_{ii}^{nn} u_i^j + \sum_{j=1}^{N_T} V_{ii}^{ns} u_{II}^j$$

and

$$u_s^i = \sum_{i=1}^{N_T} U_{ii}^{ns} p^i + \sum_{i=1}^{N_T} U_{ii}^{ss} q^i + \sum_{i=1}^{N_T} V_{ii}^{nn} u_i^i + \sum_{i=1}^{N_T} V_{ii}^{ns} u_{II}^i$$

for the displacements. According to the boundary conditions, there are $2 \times (N_T + N_c)$ linear algebraic equations for unknowns $p^j$, $q^j$ and $u_i^j$, $u_{II}^j$. It is worth noting that it is possible to use the displacement discontinuity equations on the non-cracked boundaries as well, and have all the unknowns as discontinuity displacements. However, it is found that the mixed-formulation is generally more accurate.

5 The Determination of stress Intensity Factors

The displacement discontinuities on the crack surfaces can be related to the stress intensity factors in a straightforward way. Here, a more accurate technique is used to obtain the stress intensity factors, through a so-called equivalence transformation [6].

The stresses on the crack surfaces can be obtained from the displacement discontinuities as follows:

$$\sigma_i^j = \sum_{j=1}^{N_c} \frac{2\mu u_i^j a_j}{\pi(1-\nu)[(x_j - x_i)^2 - a_j^2]} = \sum_{j=1}^{N_c} \alpha_{ij} u_i^j$$
and

$$\sigma^j_i = \sum_{j=1}^{N_e} \frac{2 \mu u^j_i a_j}{\pi(1-\nu)|[(x_j - x_i)^2 - a_j^2]} = \sum_{j=1}^{N_e} a_{ij} u^j_i$$

(14)

where $a_j$ is the half-length of element $j$. Using the crack line Green's function [1], for a crack of length $2c$, it follows:

$$K_1(\tau_c) = \frac{1}{\sqrt{\pi c}} \int_{-c}^{c} \sigma(x, \tau_c) \sqrt{\frac{c+x}{c-x}} dx = \sum_{i=1}^{N_e} \sigma^i_n F^\pm_i \sqrt{\pi c}$$

$$= \sum_{i=1}^{N_e} \sum_{j=1}^{N_e} a_{ij} F^\pm_i u^j_i \sqrt{\pi c} = \sum_{j=1}^{N_e} \lambda_j u^j_i \sqrt{\pi c}$$

$$K_{II}(\tau_c) = \frac{1}{\sqrt{\pi c}} \int_{-c}^{c} \sigma_s(x, \tau) \sqrt{\frac{c+x}{c-x}} dx = \sum_{j=1}^{N_e} \lambda_j u^j_{II} \sqrt{\pi c}$$

$$F^\pm_i = \frac{1}{\pi c} \int_{x_i-a_i}^{x_i+a_i} \sqrt{\frac{c+x}{c-x}} dx = \frac{1}{\pi} \left[ \arcsin \left( \frac{x}{c} \right) \pm \sqrt{1 - \left( \frac{x}{c} \right)^2} \right]_{x_i-a_i}$$

where $\tau_c = \theta + \epsilon 2k\pi/T, k = 0, 1, 2, \ldots M_r$. Accurate results are obtained with $\theta T = 5$ and $T = 20c/c_1$. Once the stress intensity factors $K_1(\tau_c)$ and $K_{II}(\tau_c)$ are determined, the stress intensity factors in the time domain may be obtained by using an inverse transformation such as that developed by Domills[5].

6 Numerical Examples

In order to assess the efficiency and accuracy of the proposed formulation a number of test examples were studied.

6.1 An isolated crack in an infinite sheet

The first example studied is that of an isolated crack in an infinite sheet subjected to a Heaviside load on the crack surfaces. For this problem the displacement discontinuity method was used with $N_c = 80$ and the number of sample numbers for the Laplace inversion was chosen as $M_r = 80$. The stress intensity factors are presented in figure 2 together with accurate solutions reported by Thau and Lu[8]. It is interesting to note that there are several kinks in both the $K_1$ and the $K_{II}$ curves for $c_1t/h < 8$ and that they are periodically spaced. The spacing is determined by the time for waves to travel from one crack tip to the other.

6.2 Central Crack in a rectangular sheet

The second problem considered is that of a central crack of length $2c$ in a rectangular sheet of width $2b$ and height of $2h$ as shown in figure 3. The sheet is subjected to uniform uniaxial stress $\sigma(t)$ at its ends with a Heaviside-function time dependence. The height to width ratio is chosen as $h/b = 2$ and the crack length to width ratio as $c/b = 0.24$. The Possion's ratio $\nu = 0.3$.

For this example the mixed formulation, equations (12) and (14), is used with $N_r = 15$, $N_c = 10$ and $M_r = 25$. The stress intensity factor $K_1$ obtained in this way is compared in figure 3 to those reported by Chen [7] using the finite difference method and Fedelinski et al [3] using the dual boundary element method. The agreement between the solutions is considered good.
Figure 2: Stress intensity factors for an isolated crack subjected to Heaviside load

Figure 3: The normalised stress intensity factors
7 Conclusions

An indirect boundary integral equation formulation based on the fictitious stress and displacement discontinuity method was presented in the Laplace transformed domain. The formulation was applied to crack problems subjected to dynamic loading. The stress intensity factors $K_I$ and $K_{II}$ were calculated using the crack-line Green's function and an equivalent stress method. The accuracy of the method was assessed by solving a pure mode I and a mixed-mode problem for which alternative solutions are available. The agreement between them is good.

References


