Solving plate bending problems by the analog equation method

M.S. Nerantzaki

Department of Civil Engineering, National Technical University, Zografou Campus, GR-15773 Athens, Greece

ABSTRACT

In this paper the Analog Equation Method (AEM) is employed to solve a class of plate problems whose fundamental solution either can not established or is difficult to treat numerically, such as plates resting on non-homogeneous or non-linear elastic foundation, static or dynamic behaviour of plates subjected to in-plane forces. Numerical examples are presented to illustrate the efficiency and the accuracy of the proposed method.

INTRODUCTION

The BEM solution to the biharmonic equation governing the bending of thin elastic plates is well established [1-5]. However, for many plate problems a pure boundary element method can not be developed due to the fact that the fundamental solution either can not be determined or it has accomplicate form and therefore it is difficult to treat it numerically. To cope with these problems the investigators have developed domain/boundary element methods (D/BEM's). Their use is inevitable when the governing operator includes non-linear terms. In this paper a novel BEM-based technique, the Analog Equation Method [6], is employed to solve plate problems with the aforementioned difficulties as far as the fundamental solution. According to this method the solution of the given problem is obtained by solving a linear plate bending problem described by the biharmonic operator under the same boundary conditions and subjected to an appropriate fictitious transverse load. The method is applied to both static and dynamic problems. Several example problems are solved which demonstrate the efficiency of the method.
THE ANALOG EQUATION METHOD

The class of plate problem that will be treated here are governed by the following equations

\[ D \nabla^4 w + f(w, w_{xx}, w_{xy}, w_{yy}, x, y) + \rho \ddot{w} = p \quad \text{in } \Omega \quad (1) \]

\[ \alpha_1 w + \alpha_2 V_n = \alpha_3 \quad \beta_1 w_n + \beta_2 M_n = \beta_3 \quad \text{on } \Gamma \quad (2a,b) \]

and

\[ w(x,y;0) = \bar{w}(x,y), \quad \dot{w}(x,y;0) = \ddot{w}(x,y) \quad \text{in } \Omega \quad (3) \]

where \( w = w(x,y;t) \) is the transverse deflection of the plate; \( \rho \) is the surface mass density; \( V_n, M_n \) are the effective shear force and the bending moment on the boundary respectively; \( \alpha_i, \beta_i \ (i = 1,2,3) \) are functions specified on the boundary of the plate; \( p = p(x,y;t) \) is the transverse loading on the plate; \( \bar{w}(x,y), \ddot{w}(x,y) \) are specified functions; \( D = \frac{Eh^3}{12(1-\nu^2)} \) is the flexural rigidity of the plate. Finally, \( f(w, w_{xx}, w_{xy}, w_{yy}, x, y) \) is in general a non-linear function of its arguments. The initial boundary value problem of Eqs(1),(2),(3) is solved using AEM, which in the problem at hand is applied as following.

Let \( w \) be the sought solution of the problem. If the biharmonic operator is applied to this function we have

\[ \nabla^4 w = q(x,y;t) \quad (4) \]

Eq.(4) indicates that the solution of the original problem can be obtained from the solution of a linear quasi-static plate problem with unit stiffness and subjected to the fictitious force distribution \( q(x,y;t) \) under the prescribed boundary conditions.

The problem is converted to that of establishing the unknown load density \( q(x,y;t) \). This is accomplished using BEM as following:

The solution to Eq.(4) is given in an integral form as \[3\].

\[ \epsilon w(P,t) = \int_\Omega \Lambda_4(r)qd\Omega - \int_\Gamma [\Lambda_1(r)\Omega + \Lambda_2(r)X + \Lambda_3(r)\Phi + \Lambda_4(r)\Psi]ds \quad (5) \]

Application of the Laplacian operator yields
where $\varepsilon = \pi, \pi/2$, 0 depending on whether the point $P$ is inside the domain $\Omega$, on the boundary $\Gamma$, and outside $\Omega$, respectively. Note that the boundary has been assumed to be smooth at the point $P$. The kernels $\Lambda_i = \Lambda_i(r)$, $r = |P - q|$, $P \in \Omega$, $q \in \Gamma$, are given as

\begin{align}
\Lambda_1(r) &= -\frac{\cos \varphi}{r} \\
\Lambda_2(r) &= \ell nr + 1 \\
\Lambda_3(r) &= -\frac{1}{4} (2\ell nr + r) \cos \varphi \\
\Lambda_4(r) &= \frac{1}{4} r^2 \ell nr
\end{align}

Using curvilinear co-ordinates on the boundary, that is the distance along the outward normal $n$ to the boundary and the arc length $s$, the boundary conditions (2a,b) can be written as

\begin{align}
- (\nu - 1)(X_{ss} - \kappa \Omega_s, s)_s = \alpha_3 \\
(\nu - 1)(\Phi_{ss} + \kappa X) = \beta_3
\end{align}

where $\kappa = \kappa(s)$ is the curvature of the boundary. In Eqs(5),(6),(8),(9) the following notation has been used

\begin{align}
\Omega = w(s,t) \\
X = \frac{\partial}{\partial n} w(s,t) \\
\Phi = \nabla^2 w(s,t) \\
\Psi = \frac{\partial}{\partial n} \nabla^2 w(s,t)
\end{align}
solved using the boundary element technique. Constant boundary element
have been employed with parabolic approximation of its geometry. The
domain integrals are evaluated using constant triangle or rectangular cells.

The above discretization yields the following set of linear equations.

\[
\begin{bmatrix}
[A_{11}] & [A_{12}] & [0] & [A_{14}] \\
[A_{21}] & [A_{22}] & [A_{23}] & [0] \\
[A_{31}] & [A_{32}] & [A_{33}] & [A_{34}] \\
[0] & [0] & [A_{43}] & [A_{44}]
\end{bmatrix}
\begin{bmatrix}
\{\Omega\} \\
\{X\} \\
\{\Phi\} \\
\{\Psi\}
\end{bmatrix}
= 
\begin{bmatrix}
\{B_1\} \\
\{B_2\} \\
\{0\} \\
\{0\}
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
[C_3] \\
[C_4]
\end{bmatrix}
\{q\}
\]  

(11)

where \([A_{ij}] (i, j = 1, 2, 3, 4)\) are \(N \times N\) known coefficient matrices,
\([B_i] (i = 1, 2)\) \(N \times 1\) known constant column matrices and \([C_i] (i = 3, 4)\)
\(N \times M\) known coefficient matrices. \(\Omega, X, \Phi, \Psi\) are \(N \times 1\) vectors
including the nodal values of the unknown boundary quantities, while \(\{q\}\) is an \(M \times 1\) vector including the nodal values of the unknown fictitious loading
at the nodal points inside \(\Omega\). \(N\) is the number of the boundary nodal points,
whereas \(M\) is the number of the domain nodal points.

Using Eqs(1) boundary quantity \(\Omega, X, \Phi, \Psi\) can be eliminated from the
discretized counter part of Eq.(5), which after collocation at the \(M\) domain
nodal points yields

\[
\{w\} = [G]\{q\}
\]  

(12)

where \(\{w\}\) is an \(M \times 1\) vector including the values of the function \(w\) at the \(M\)
domain nodal points and \([G]\) is an \(M \times M\) known coefficient matrix.

Subsequent differentiation of equation (5) twice with respect to \(x\) and \(y\)
yields

\[
w_{xx}(P) = \int_{\Omega} (\Lambda_4)_{xx} q d\Omega \\
- \int_{\Gamma} [(\Lambda_1)_{xx} \Omega + (\Lambda_2)_{xx} X + (\Lambda_3)_{xx} \Phi + (\Lambda_4)_{xx} \Psi) ds
\]  

(13a)

\[
w_{yy}(P) = \int_{\Omega} (\Lambda_4)_{yy} q d\Omega \\
- \int_{\Gamma} [(\Lambda_1)_{yy} \Omega + (\Lambda_2)_{yy} X + (\Lambda_3)_{yy} \Phi + (\Lambda_4)_{yy} \Psi) ds
\]  

(13b)
The derivatives of the kernels are given in Ref. [7]. Eliminating of the boundary quantities from the discretized counterparts of equations (13a, b, c) by means of equations (11) and collocating at the $M$ nodal points inside $\Omega$ yields

\[
\begin{align*}
\{ w_{xx} \} &= [G_{xx}]\{q \} \\
\{ w_{yy} \} &= [G_{yy}]\{q \} \\
\{ w_{xy} \} &= [G_{xy}]\{q \}
\end{align*}
\] (14) (15) (16)

where $[G_{xx}],[G_{yy}],[G_{xy}]$ are known $M \times M$ coefficient matrices. Note that equations (12), (14)-(16) are valid for homogeneous boundary conditions ($\alpha_3 = \beta_3 = 0$). For non-homogeneous boundary conditions an additive constant vector will appear in the right hand side of these equations.

The final step of AEM is to apply equations (1) to the $M$ nodal points inside the domain $\Omega$. Using equations (12), (14)-(16), we obtain

\[
[M]\{\ddot{q}\} + D\{q\} + \{f(q)\} = \{p\} 
\] (17)

Applying Eq.(12) and its derivative with respect to time at $t = 0$, the initial conditions for Eq.(17) are determined as

\[
\begin{align*}
\{q\} &= [G]^{-1}\{\bar{w}\} \\
\{\dot{q}\} &= [G]^{-1}\{\dot{w}\}
\end{align*}
\] (18a) (18b)

The equation for the static problem is obtained from Eq.(17) by neglecting the inertia term. Thus

\[
D\{q\} + \{f(q)\} = \{p\} 
\] (19)

Eq.(17) can be solved numerically using a direct time step integration scheme for non-linear equations, while Eq.(19) can be solved by the fix-point method as presented in Ref.[8].
NUMERICAL RESULTS

On the basis of analysis presented in previous sections a computer program has been written and some example plates have been studied to illustrate the efficiency of the developed method and investigate its accuracy.

Example 1

A circular clamped plate having radius \( \alpha \) and resting on Winkler foundation with stiffness \( k = D \) has been studied. For this problem, Equations (1) and (2) are written as

\[
\begin{align*}
D\nabla^4 w + Dw &= p \\
w &= 0 \\
w_n &= 0
\end{align*}
\]

(20)

(21)

In this case it is \( f = Dw \).

The obtained numerical results for the deflection and the radial moment along the radius are presented in Table 1. They are good agreement with those obtained by an analytical solution.

Table 1. Deflections and radial bending moments in a uniformly loaded clamped circular plate on Winkler elastic foundation \( (f = Dw, \ \nu = 0.3) \).

<table>
<thead>
<tr>
<th>Position</th>
<th>( \bar{w} = w / (p\alpha^4 / D) )</th>
<th>( \bar{M}_r = M_r / pa^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r / \alpha )</td>
<td>AEM</td>
<td>Analytic [8]</td>
</tr>
<tr>
<td>.0985</td>
<td>.1838E-2</td>
<td>.1838E-2</td>
</tr>
<tr>
<td>.3045</td>
<td>.1701E-2</td>
<td>.1703E-2</td>
</tr>
<tr>
<td>.562</td>
<td>.1220E-2</td>
<td>.1219E-2</td>
</tr>
<tr>
<td>.802</td>
<td>.4493E-3</td>
<td>.4340E-3</td>
</tr>
<tr>
<td>.9602</td>
<td>.2910E-4</td>
<td>.2481E-4</td>
</tr>
</tbody>
</table>

Example 2

A circular simply supported plate having radius \( \alpha \) and resting on Winkler foundation with stiffness \( k = D \) has been studied. Equation (20) describes also this problem with \( w = 0 \) and \( Mn = 0 \) on the boundary. The obtained numerical results for the deflection and the radial moment along the radius are presented in Table 2.

Example 3

A circular clamped plate having radius \( \alpha \) and resting on a two-parameter (Pasternak) elastic foundation with \( k = D \) and
Table 2. Deflections and radial bending moments in a uniformly loaded simply supported circular plate on Winkler elastic foundation \((f = Dw, \; \nu = 0.3)\).

<table>
<thead>
<tr>
<th>Position</th>
<th>(w = w / (p\alpha^4 / D))</th>
<th>(M_r = M_r / p\alpha^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r / \alpha)</td>
<td>(AEM)</td>
<td>(Analytic [8])</td>
</tr>
<tr>
<td>.0985</td>
<td>.1858E-2</td>
<td>.1858E-2</td>
</tr>
<tr>
<td>.3045</td>
<td>.1845E-2</td>
<td>.1846E-2</td>
</tr>
<tr>
<td>.802</td>
<td>.9838E-3</td>
<td>.9769E-3</td>
</tr>
<tr>
<td>.9602</td>
<td>.2186E-3</td>
<td>.2136E-3</td>
</tr>
</tbody>
</table>

\(G = \beta D\) \((\beta = 0.1, 0.5, 1.0, 3.0)\). For this problem Eqs(1) and (2) are written as

\[
DV^4w - GV^2w + kD = p
\]  
(22)

\[
w = 0 \quad w', n = 0
\]  
(23)

In this case it is \(f = -GV^2w + kD\).

The obtained numerical results for the deflection and the radial moment along the radius are presented in Table 3.

Table 3. Deflections \(\bar{w} = w / (p\alpha^4 / D)\) and radial bending moments \(\bar{M}_r = M_r / p\alpha^2\) in a uniformly loaded clamped circular plate on two-parameter elastic foundation \((f = -GV^2w + kw, \; \nu = 0.3, k = D, \; G = \beta D)\)

| \(\beta\) | Radial position \(r / \alpha\) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\bar{w}\) | .9602 | .802 | .562 | .3045 | .0985 |
| .1 | -.2921E-4 | .4431E-3 | .1195E-2 | .1664E-2 | .1799E-2 |
| \(\bar{M}_r\) | -.2801E-1 | -.2469E-2 | .7558E-2 | .5934E-2 | .3925E-2 |
| .5 | .2976E-4 | .4213E-3 | .1106E-2 | .1535E-2 | .1662E-2 |
| \(\bar{M}_r\) | -.2743E-1 | -.1684E-2 | .6835E-2 | .5354E-2 | .3738E-2 |
| 1.0 | .3061E-4 | .3969E-3 | .1014E-2 | .1400E-2 | .1519E-2 |
| \(\bar{M}_r\) | -.2685E-1 | -.9560E-3 | .6088E-2 | .4779E-2 | .3522E-2 |
| \(\bar{M}_r\) | -.2647E-1 | .4654E-3 | .4147E-2 | .3377E-2 | .2835E-2 |
Example 4

The free vibrations of a simply supported square plate with side length $a$, subjected to a uniformly distributed in-plane load $N_x$ along the edges $x = \pm a / 2$ has been studied for various types of in-plane boundary conditions. For this problem Equations (1) and (2) are written as

\begin{align}
D \nabla^4 w - N_x w_{xx} - \rho \omega^2 w &= 0 \quad (24) \\
w &= 0 \quad M_n = 0 \quad (25)
\end{align}

In this case Equation (17) becomes

\[(D - [N_x][G_{xx}] - \rho \omega^2 [G])\{q\} = 0 \quad (26)\]

and the frequency equations is

\[
\det[D - [N_x][G_{xx}] - \rho \omega^2 [G]] = 0
\]

Fig.1. Fundamental frequency parameter $\lambda = \alpha^4 \sqrt{\omega^2 \rho / D}$ versus the in-plane load $\bar{N}_x = N_x \alpha^2 / \pi^2 D$ at $x = \pm \alpha / 2$ in a simply supported square plate for various types of boundary conditions along the edges $y = \pm \alpha / 2$, $(f = -N_x w_{xx} - \rho \omega^2)$. Curve I: $N_{yx} = v = 0$; Curve II: $u = v = 0$; Curve III: $N_y = N_{yx} = 0$; Curve IV: $u = N_y = 0$. 
where \([N_x]\) is diagonal matrix including the nodal values of \(N_x\) in the interior of the plate. They are obtained from the solution for the linear elastostatic problem. In Fig.1 the dependence of the frequency parameter \(\lambda\) is plotted versus the dimensionless in-plane load on the basis of results obtained form the numerical evaluation of the frequency equation. The linear buckling load is obtained for \(\omega = 0\).

REFERENCES