Trefftz boundary formulation for obtaining generalized eigenvalue problem of the Helmholtz equation

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Abstract

The well-known Trefftz scheme is applied to the formulation of the Helmholtz equation for eigenvalue determination. Both direct and indirect methods are employed for two-dimensional problems with homogeneous Dirichlet, Neumann and mixed boundary conditions. The unknown eigenvalue included implicitly in the T-complete series of Trefftz functions is separated by their series expansion. The final eigenvalue problem is formulated as of the generalized-type, which can be solved by using existing subroutine as a blackbox.

Introduction

The Trefftz method, according to classification of numerical methods by Collatz[1], belongs to the boundary-type solutions; i.e., using a series of functions satisfying the governing field equation of the given boundary-value problem, the solution is constructed to satisfy the boundary conditions[2]. Consequently we do not need any discretization of the domain under consideration, say, by finite elements, which is opposite to the domain-type solutions like finite element and finite difference methods. For fulfillment of the boundary conditions, collocation and certain weighted residuals schemes are typical among others. Marked merit of the Trefftz method is less dimensionality mentioned similarly to the boundary element method.

In this paper, we present a new method of eigenvalue determination of the Helmholtz equation following the Trefftz-type formulation. The principal objective is to develop the scheme offered as the “generalized eigenvalue problem” without discretization of the domain. It has generally been believed that
Boundary Element Method XVI

eigenvalue determination by the boundary element method is not necessarily efficient, while so far some improvements have been reported [3-9]. In what follows, a distinct boundary method using the Trefftz scheme will be reported.

Trefftz method

Consider the following Helmholtz equation in the two-dimensional domain Ω bounded by the boundary Γ:

\[ \nabla^2 u + k^2 u = 0 \]  

where \( u \) is the potential and \( k \) is the wavenumber, unknown for the eigenvalue problem. The corresponding boundary conditions, the homogenous Dirichlet and Neumann types, are offered as:

\[
\begin{align*}
  u &= 0 \quad \text{(on } \Gamma_1) \\
  \frac{\partial u}{\partial n} &= 0 \quad \text{(on } \Gamma_2)
\end{align*}
\]  

where combination of \( \Gamma_1 \) and \( \Gamma_2 \) is the entire boundary \( \Gamma \), and \( n \) denotes the outward unit normal taken on the boundary.

The following two schemes are adopted here for formulating the problem given by Eqs.(1) and (2).

Collocation method:

The function \( u \) is assumed to be expressed as

\[ u = \sum_i N_i a_i \]  

where \( N_i \) is a series of the functions satisfying the original differential equation (1),

\[ \nabla^2 N_i + k^2 N_i = 0 \]  

\( N_i \), referred to C-complete or T-complete functions for the Helmholtz equation, is represented for the bounded domain as,

\[ \{ N_i \} = \{ J_\theta (kr), J_i (kr) \cos (i\theta), J_i (kr) \sin (i\theta) \} \quad (i = 1, 2, \cdots) \]  

where \( J_i \) is the \( i \)th-order Bessel function of the first kind and \( r \) and \( \theta \) are the plane polar coordinates in the two-dimensional space, whose origin and reference axis are taken arbitrarily.

\( a_i \)'s in Eq.(3) are constant coefficients to be determined to satisfy the boundary condition (2). For the fulfillment, selected points (collocation points) \( p_j \)'s, \( N_1 \) on the boundary \( \Gamma_1 \) and \( N_2 \) on the boundary \( \Gamma_2 \) (total \( N \) points \( N_1 + N_2 = N \)), are taken and the following equations are obtained:

\[
\begin{align*}
  \sum_i N_i (p_j) a_i &= 0 \quad (j = 1, 2, \cdots, N_1) \\
  \sum_i \frac{\partial N_i}{\partial n} (p_j) a_i &= 0 \quad (j = N_1 + 1, \cdots, N)
\end{align*}
\]
from which the eigenvalue problem is formulated as

$$[M]\{a\} = \{0\}$$  \hspace{1cm} (7)

where the components $m_{ij}$ of $[M]$ of $N$-order are

$$m_{ij} = \begin{cases} N_i(p_j) & (i = 1, 2, \cdots, N; \ j = 1, 2, \cdots, N) \\ \frac{\partial N_i}{\partial n}(p_j) & (i = 1, 2, \cdots, N; \ j = N_1 + 1, \cdots, N) \end{cases}$$  \hspace{1cm} (8)

$\{a\}$ in Eq.(7) is the vector with its components $a_i(i = 1, 2, \cdots, N)$.

Due to the nonvanishing condition of the vector $\{a\}$, we obtain from Eq.(7)

$$\text{det}[M] = 0$$  \hspace{1cm} (9)

Since the components of the matrix $[M]$ contain the unknown $k$ as assumed in Eq.(5), the ordinary routine for eigenvalue determination cannot be easily applicable.

**Weighted residual method:**

In place of the collocation presented above, we may employ the following weighted residual expression using the weight function $N_i$, same to the above:

$$\int_{\Omega} (\nabla^2 u + k^2 u)N_i d\Omega = 0$$  \hspace{1cm} (10)

Integrating Eq.(10) twice by parts, we obtain, after substitution of Eq.(2),

$$\int_{\Gamma_1} \frac{\partial u}{\partial n}N_id\Gamma = \int_{\Gamma_2} u \frac{\partial N_i}{\partial n} d\Gamma$$  \hspace{1cm} (11)

In order to compute Eq.(11), we employ the simplest elements, piecewise constant interpolations; $L_1$ elements on $\Gamma_1$ and $L_2$ on $\Gamma_2$ ($L_1 + L_2 = L$). Then, Eq.(11) becomes as

$$\sum_{j=1}^{L_1} \left( \frac{\partial u}{\partial n} \right)_j \int_{\Gamma_j} N_id\Gamma = \sum_{j=L_1+1}^{L} u_j \int_{\Gamma_j} \frac{\partial N_i}{\partial n} d\Gamma$$  \hspace{1cm} (12)

where $\Gamma_j$ stands for the $j$th element.

One equation holds for each $N_i$, and thus we take $L$ weight functions $N_1, \cdots, N_L$ and define the following vector $\{x\}$:

$$\{(\partial u/\partial n)_1, (\partial u/\partial n)_2, \cdots, (\partial u/\partial n)_{L_1}, u_{L_1+1}, \cdots, u_L\}^T = \{x\}$$  \hspace{1cm} (13)

A system of $L$ equations constitutes the following eigenvalue problem:

$$[G, \ -H]\{x\} = \{0\}$$  \hspace{1cm} (14)

where matrices $[G]$ and $[H]$ are of order $L \times L_1$ and $L \times L_2$ respectively; their elements $g_{ij}$ and $h_{ij}$ are given as

$$g_{ij} = \int_{\Gamma_j} N_i d\Gamma$$  \hspace{1cm} (15)

$$h_{ij} = \int_{\Gamma_j} \frac{\partial N_i}{\partial n} d\Gamma$$
For the latter formulation, Eq.(14), it seems also difficult to determine eigenvalues from the proper equation

\[ \det \left[ G, - H \right] = 0 \]  

(16)

because the elements include the unknown \( k \) implicitly.

**Expression by lambda matrices**

The Bessel function appearing in the expression(5) is represented as a series with respect to \( k \),

\[ J_i(kr) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+i)!} k^{i+2m} \left( \frac{r}{2} \right)^{i+2m} \]  

(17)

or

\[ J_i(kr) = k^i \sum_{m=0}^{\infty} \frac{(1-k^2)^m}{m!} \left( \frac{r}{2} \right)^m J_{i+m}(r) \]  

(18)

the latter series converges faster than the former.

Substituting Eq.(18) into Eq.(6), the components of the matrix \([M]\) become

\[ m_{ij} = \left\{ \begin{array}{ll}
\sum_{m=0}^{\infty} \frac{(1-k^2)^m}{m!} \left( \frac{r_j}{2} \right)^m J_{i+m} \cos(i \theta_j) & (j = 1, 2, \cdots, N_1) \\
\sum_{m=0}^{\infty} \frac{(1-k^2)^m}{m!} \left( \frac{r_j}{2} \right)^m J_{i+m} \sin(i \theta_j) & (j = 1, 2, \cdots, N_1)
\end{array} \right. \]  

(19)

\[ m_{ij} = \left\{ \begin{array}{ll}
k^i \sum_{m=0}^{\infty} \frac{(1-k^2)^m}{m!} \left( \frac{r_j}{2} \right)^m \left\{ \frac{m}{r_j} J_{i+m}(r_j) + J'_{i+m}(r_j) \right\} \cos(i \theta_j) & (j = 1, 2, \cdots, N_1) \\
-iJ_{i+m}(r_j) \sin(i \theta_j) & (j = N_1 + 1, \cdots, N)
\end{array} \right. \]  

(20)

where \( r_j \) and \( \theta_j \) are \( r \) and \( \theta \) coordinates of the \( j \)th collocation point \( p_j \).

We can notice that, from Eqs.(19) and (20), these elements are series with respect to \( 1-k^2 \) except a multiplier \( k^i \) (\( k^i \) can be omitted when they are inserted in the expression of the determinant, Eq.(9)),

\[ [M] = [M^0] + (1-k^2)[M^1] + (1-k^2)^2[M^2] + \cdots \]  

(21)

where \([M^i]\) does not include \( k \) and \([M]\) is referred to the polynomial matrix or lambda matrix.

Another substitution of Eq.(18) into Eq.(15) yields the components of the matrices \([G]\) and \([H]\) as

\[ g_{ij} = \left\{ \begin{array}{ll}
\sum_{m=0}^{\infty} (1-k^2)^m \int_{\Gamma_j} \frac{J_{i+m}(r)}{m!} \left( \frac{r}{2} \right)^m \cos(i \theta) d\Gamma \\
\sum_{m=0}^{\infty} (1-k^2)^m \int_{\Gamma_j} \frac{J_{i+m}(r)}{m!} \left( \frac{r}{2} \right)^m \sin(i \theta) d\Gamma
\end{array} \right. \]  

(22)
where common coefficient $k^i$ has been omitted in Eqs.(22) and (23). These components are also the polynomial matrices with respect to $1 - k^2 (= k^i)$

$$\begin{align*}
\mathbf{G} &= \mathbf{G}^0 + k'[\mathbf{G}^1] + k''[\mathbf{G}^2] + \cdots \\
\mathbf{H} &= \mathbf{H}^0 + k'[\mathbf{H}^1] + k''[\mathbf{H}^2] + \cdots 
\end{align*}$$

As shown above, the coefficient matrices of the nonvanishing vector in the eigenvalue problem, formulated using two distinct schemes, collocation and weighted residual, can be represented as the polynomial matrices. Similar formulation can be possible for the three-dimensional problem[10].

Formulation as the generalized eigenvalue problem

The most conventional approach for determination of eigenvalue in virtue of Eq.(9) or (14) is direct determinant search; i.e., to search the zeros (in practice, sufficiently small values) of the determinant from its distribution computed with small increment of $k$ [8]. Other method relies on the Newton iteration[9]. These methods generally require a huge amount of complicated computation, and therefore they are computationally inefficient while the solution space is confined only on the boundary.

As shown in the previous section, the coefficient matrices in Eqs.(7) and (14) are represented as the polynomial matrices in terms of the eigenvalue. It can be said, therefore, that even conventional determination schemes may raise computational efficiency. We here, however, devise a different approach with help of the polynomial matrix representation by converting to the generalized eigenvalue problem, for which computation with existing subroutines is available.

First, we denote Eqs.(7) and (14) as follows:

$$\begin{align*}
\mathbf{A}\{y\} &= \{0\} \tag{25}
\end{align*}$$

were

$$\begin{align*}
\mathbf{A} &= \mathbf{A}^0 + k'[\mathbf{A}^1] + k''[\mathbf{A}^2] + \cdots 
\end{align*}$$

The following substitutions correspond to each equation: for Eq.(7)

$$\begin{align*}
\mathbf{A} &= \mathbf{M}, \quad \{y\} = \{a\} \tag{27}
\end{align*}$$

for Eq.(14)

$$\begin{align*}
\mathbf{A} &= \mathbf{G} - \mathbf{H}, \quad \{y\} = \{x\} \tag{28}
\end{align*}$$
Confining up to the term $[\mathbf{A}^m]$ of the infinite series $[\mathbf{A}]$, we obtain a vector series

$$\{y^i\} = k^i \{y^{i-1}\} \quad (i = 1, \ldots, m)$$  \hspace{1cm} (29)

where

$$\{y^0\} = \{y\}$$  \hspace{1cm} (30)

Substitution of Eqs.(26) and (29) into Eq.(25) leads us to

$$[\mathbf{A}^0]\{y^0\} + [\mathbf{A}^1]\{y^1\} + \cdots + [\mathbf{A}^m]\{y^m\} = \{0\}$$  \hspace{1cm} (31)

Now, we define a vector series

$$\{\tilde{y}\} = \{\{y^0\}\{y^1\}\cdots\{y^{m-1}\}\}^T$$  \hspace{1cm} (32)

and matrices $[\tilde{\mathbf{A}}]$ and $[\tilde{\mathbf{B}}]$ as follows:

$$[\tilde{\mathbf{A}}] = \begin{bmatrix}
[\mathbf{A}^0] & [\mathbf{A}^1] & \cdots & [\mathbf{A}^{m-1}]
0 & [\mathbf{I}] & 0 & \cdots & 0 \\
\vdots & [\mathbf{I}] & 0 & \ddots & 0 \\
0 & \ddots & 0 & \cdots & 0 \\
0 & \cdots & 0 & [\mathbf{I}]
\end{bmatrix}$$  \hspace{1cm} (33)

$$[\tilde{\mathbf{B}}] = \begin{bmatrix}
0 & 0 & \cdots & -[\mathbf{A}^m] \\
0 & [\mathbf{I}] & 0 & \cdots & 0 \\
\vdots & [\mathbf{I}] & 0 & \ddots & 0 \\
0 & \cdots & 0 & [\mathbf{I}] & 0
\end{bmatrix}$$  \hspace{1cm} (34)

consequently, Eq.(31) is transformed as

$$[\tilde{\mathbf{A}}]\{\tilde{y}\} = k' [\tilde{\mathbf{B}}]\{\tilde{y}\}$$  \hspace{1cm} (35)

which is known as the generalized eigenvalue problem. The matrices $[\tilde{\mathbf{A}}]$ and $[\tilde{\mathbf{B}}]$ are free from the unknown $k$ and can be constructed easily by $[\mathbf{A}^i]$ alone. Eigenvalues are determined from Eq.(35) by using existing subroutine as a blackbox, which does not require initial approximation and iteration conducted by user.

**Numerical examples**

The proposed method is examined by some numerical examples, compared with known analytical rigorous solution. As the solver for the generalized eigenvalue problem, scientific subroutine offered from Nagoya University Computer Center is employed through subroutine call. Piecewise constant boundary elements are taken for the weighted residual scheme. Collocation points are placed on the just centroid of the boundary elements while they are not required.
Example 1: Unit circle with $\partial u/\partial n = 0$ along boundary

The rigorous solution is given as

$$k_e = \mu_{s,t}$$

(36)

where, $\mu_{s,t}$ stands for $t$th solution of $\frac{d}{d\mu} J_s(\mu) = 0$ ($J_s$ is $s$-th order Bessel function of the first kind). $s$ and $t$ are mode numbers in the radial and circumferential directions.

The corresponding numerical solution by the collocation method is shown and compared in Table 1. Numbers of terms of series expansion of the Bessel function and of collocation points distributed equal distance along boundary are $m = 10$ and $L = 20$, respectively. Good agreement is obtained in the quadruple precision computation.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t$</th>
<th>analytical</th>
<th>present (error %)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0</td>
<td>1.841</td>
<td>1.841 (0.00)</td>
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<tr>
<td>2</td>
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<td>3.832</td>
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<tr>
<td>3</td>
<td>0</td>
<td>4.201</td>
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</tr>
<tr>
<td>4</td>
<td>0</td>
<td>5.318</td>
<td>5.318 (0.00)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5.331</td>
<td>5.331 (0.00)</td>
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<tr>
<td>5</td>
<td>0</td>
<td>6.416</td>
<td>6.416 (0.00)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>6.706</td>
<td>6.708 (0.03)</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>7.016</td>
<td>6.979 (0.52)</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>7.501</td>
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<tr>
<td>3</td>
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<td>8.015</td>
<td>8.189 (2.17)</td>
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<tr>
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Table 2. Comparison of eigenvalues by weighted residual method (Prob. 2)

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<th>Weighted residual (error %)</th>
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<td>16 elements</td>
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<td>2</td>
<td>0</td>
<td>6.28</td>
<td>6.28 (0.00)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8.18</td>
<td>8.18 (0.00)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>9.42</td>
<td>9.45 (0.03)</td>
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<tr>
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<td>1</td>
<td>10.78</td>
<td>10.72 (0.55)</td>
</tr>
<tr>
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<tr>
<td>3</td>
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Fig. 1 Problem 2.

Fig. 2 Element discretizations

Fig. 3 Collocation points
Table 3. Comparison of eigenvalues by collocation method (Prob. 2)

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<th>Analytical</th>
<th>Collocation (error %)</th>
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</tr>
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<td>9.32 (1.06)</td>
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<td>10.78</td>
<td>10.73 (0.46)</td>
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<td>10.87 (0.55)</td>
</tr>
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<td>12.19 (0.02)</td>
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<td>0</td>
<td>12.57</td>
<td>12.54 (0.02)</td>
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<tr>
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<td>1</td>
<td>13.61</td>
<td>13.23 (2.79)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>14.09</td>
<td>-</td>
</tr>
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</table>

Example 2: Rectangle with mixed boundary condition

Figure 1 shows the second example with geometry and prescribed boundary conditions, of which analytical solution is, for the mode numbers in x and y directions s and t,

\[ k_x = \pi \sqrt{\left(\frac{s}{L_x}\right)^2 + \left(\frac{t}{L_y}\right)^2} \quad (s \geq 1, t \geq 0) \quad (37) \]

Boundary discretizations employed for the weighted residual scheme and collocation points for the collocation scheme are respectively shown in Figs. 2 and 3. Results obtained for \( m = 10 \) are depicted in Tables 2 and 3. Both schemes give highly accurate solutions and the more elements or collocation points, the higher the solution accuracy. Computer time required for collocation method is about one-sixth for the weighted residual. The above computations are carried out with quadruple precision.

Concluding remarks

Using the T-complete series of the Trefftz method and collocation and/or weighted residual, we formulated the eigenvalue problem for the two-dimensional Helmholtz equation. The problem was successfully transformed to the generalized eigenvalue problem and could be solved through existing subroutine. The present methods can determine eigenvalues very efficiently by discretizing boundary alone or by taking collocation points on the boundary. Collocation scheme does not require any integral computations, while weighted residual scheme has its possibility of modification of accuracy by higher-order boundary elements.
References


