Combined Laplace transform and dual reciprocity method for solving time-dependent diffusion equations with nonlinear source terms

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Abstract

In this paper, a numerical method with the combined Laplace transform and the Dual Reciprocity Method proposed by Zhu et al. [1] is extended to solve time-dependent diffusion equations with nonlinear source terms. The nonlinear source terms are linearised first and then an iteration is carried out until a convergent value of the unknown function at a desired time level is obtained. We shall demonstrate, through the preliminary results of our numerical experiments, that the method allows an accurate numerical solution at any desired observation time to be calculated directly and efficiently. It is believed that such a high efficiency is achieved as a direct consequence of the fact that no domain integral is involved in the calculation and the time variable is also temporarily removed from the problem.

1 Introduction

Time-dependent diffusion equations have many applications in practice; many important problems in engineering and industry can be modelled by a time-dependent diffusion equation. Examples in the literature include, among many others, microwave heating process [2], spontaneous ignition [3] and mass transport in groundwater [4]. Efficiently and accurately solving time-dependent diffusion equations, especially nonlinear ones, are of great concern to applied mathematicians and engineers.

Linear diffusion equations are relatively easier to be solved compared with nonlinear ones; many highly accurate and efficient schemes have been proposed. For example, problems governed by a linear diffusion equation can be solved by a finite difference scheme [5], a finite element scheme [6] or a boundary element scheme [7]. In 1993, based on the elegant dual reciprocity method (DRM) proposed by Nardini and Brebbia [8] to convert
a singular volume integral into boundary integrals, Zhu et al. [1] combined
the Laplace transform and the DRM into what they called the LTDRM
(Laplace transform dual reciprocity method) to solve linear time-dependent
diffusion equations. They showed, through their numerical experiments,
that highly accurate results were obtained in a very efficient manner. Such
a high accuracy and efficiency rely on the fact that neither domain integral
nor time marching scheme is involved in a calculation. Alternatively, one
can say that the elegance of the LTDRM is that it reduces the dimension
of the problems by two. Unfortunately, at that time, the LTDRM could
only be applied to linear cases; the application to nonlinear cases was not
successful.

The extension of the LTDRM to nonlinear cases is far from trivial. First
of all, a successful performance of the Laplace transform is crucial for the
method to be applied. However, as well known, the Laplace transform can
be only performed to linear governing equations. Therefore, to cleverly con-
struct a linearisation of the governing equations and an iterative process is a
challenge and a key to the success of the extension of the LTDRM to nonlin-
ear cases. In this paper, we shall demonstrate a successful extension of the
LTDRM to solving nonlinear time-dependent diffusion equations by solving
a sample governing diffusion equation with a nonlinear source term which
was used by Zhu et al. [2] in modelling the microwave heating of a square
slab. The results obtained from using the extended LTDRM are compared
with those provided by Zhu et al. [1] who used a time marching scheme; the
efficiency and the accuracy of the LTDRM can thus be demonstrated. As
the method we shall propose here is fairly general, the extended LTDRM
can be easily applied to other types of nonlinear diffusion equations.

2 The Extended LTDRM

The time-dependent diffusion equation describing a microwave heating pro-
cess with a nonlinear source term, in the power-law form, can be written as

\[
\nabla^2 u(x, t) = \frac{\partial u(x, t)}{\partial t} - \beta g(x)u^n(x, t), \quad x \in \Omega,
\]

where \( n \) and \( \beta \) are given numbers, \( g \) is an arbitrary function, \( x = (x, y) \)
is a spatial variable and \( \Omega \) is the domain to be considered. The problem is
subject to boundary conditions of the form

\[
\begin{align*}
    u(x, t) &= \overline{u}(x, t), \quad x \in \Gamma_1, \\
    q(x, t) &= \frac{\partial u(x, t)}{\partial n(x)} = \overline{q}(x, t), \quad x \in \Gamma_2,
\end{align*}
\]

and initial conditions at time \( t_0 \)

\[
u(x, t) = u_0(x, t_0), \]

where \( q \) is the flux, \( n(x) \) is the unit outward normal at point \( x \), \( \overline{u}, \overline{q} \) and \( u_0 \)
are given functions and \( \Gamma_1 \cup \Gamma_2 \) is the boundary of \( \Omega \).
If \( n = 0 \) or \( 1 \), eqn (1) is linear and the LTDRM proposed by Zhu et al. [1] can be easily applied. However, a great deal of difficulty was encountered in our initial attempt of extending the LTDRM to solve the differential system (1) - (4) when \( n \geq 2 \) or the governing equation becomes nonlinear. It is well known that constructing some sort of linearisation and then performing suitable iterations are unavoidable in dealing with a nonlinear differential system. However, if we simply linearise the nonlinear term by rewriting \( u^n(x, t) \) into \( u_{(m-1)}^{n-1}(x, t)u_m(x, t) \), in which subscripts indicate the level of iteration, the Laplace transform cannot be performed. Our progress was essentially halted for quite a long time as we did not know how to circumvent this difficulty.

The halt of our progress remained until we realised that a great advantage of the LTDRM was to seek for solution at a particular observation time, especially at a large observation time [1], [9]. If the solution of the unknown function is to be sought at a particular time, say \( t_1 \), eqn (1) can be written into

\[
\nabla^2 u_m(x, t) = \frac{\partial u_m(x, t)}{\partial t} - \beta g(x) u_{(m-1)}^{n-1}(x, t_1)u_m(x, t),
\]

in which the subscripts again indicate the level of iteration. Now the Laplace transform can be readily performed on \( u_m(x, t) \) since the governing equation is linear as far as \( u_{(m-1)}^{n-1}(x, t) \) is concerned. The solution for the unknown function \( u_m(x, t) \) at the time level \( t_1 \) is obtained once the difference between \( u_{(m-1)}^{n-1}(x, t_1) \) and \( u_m(x, t) \) at \( t = t_1 \) is sufficiently small.

Once the major obstacle is removed, the rest of work is quite standard in light of the LTDRM described by Zhu et al. [1] and Satravaha et al. [9]. Since the details of the LTDRM has been clearly given in References [1] and [9], only the part relevant to this work is given below.

After applying the Laplace transform with respect to \( t \), eqn (5) becomes

\[
\nabla^2 U = \left\{ p - \beta g(x) u_{(m-1)}^{n-1}(x, t_1) \right\} U - U_0,
\]

which is subject to the boundary conditions

\[
U(x, p) = \bar{U}(x, p), \quad x \in \Gamma_1, \tag{7}
\]
\[
Q(x, p) = \frac{\partial U(x, p)}{\partial n(x)} = \bar{Q}(x, p), \quad x \in \Gamma_2, \tag{8}
\]

where \( p \) is the Laplace parameter and the subscripts attached to the Laplace transformed variables have been dropped for simplicity.

Then, upon applying the DRM in the Laplace space, the final matrix equation can be derived as

\[
H\bar{U} - G\bar{Q} = (H\bar{U} - G\bar{Q})F^{-1} \left[ \left\{ p - \beta g(x) u_{(m-1)}^{n-1}(x, t_1) \right\} U - U_0 \right]_i, \tag{9}
\]

where the subscript \( i \) denotes nodal values and all matrices are of conventional sense. By defining

\[
S = (H\bar{U} - G\bar{Q})F^{-1}, \tag{10}
\]
eqn (9) can be written in the form

\[(H - \rho S + \beta ST)\bar{U} = G\bar{Q} - S\bar{u}_0, \tag{11}\]

where \(T\) is a diagonal matrix whose diagonal elements being nodal values of \(g(x)u_{n-1}^{m-1}(x, t_1)\). This linear and non-transient system can now be solved. The solutions obtained are in the Laplace space, the Stehfest’s algorithm [10] is chosen in this study to perform a numerical inversion of the Laplace transform, the details of which have been given in References [1] and [9]. Once the inverse Laplace transform is performed, numerical values of \(u_{n-1}^{m-1}(x, t_1)\) are obtained.

In all of our numerical experiments, the iteration was performed in such a way that whenever a new solution was obtained, a stopping criteria

\[\left|\frac{(u_{n-1}^{m-1} - u_{n}^{m-1})}{(u_{n-1}^{m-1} + u_{n}^{m-1})}\right| < \varepsilon, \tag{12}\]

with which the accuracy of the final solution is controlled by a pre-set small number \(\varepsilon\), was checked at all nodes. If (12) was not satisfied, \(u_{n-1}^{m-1}(x, t_1)\) at every node was replaced by the corresponding new value \(u_{n}^{m-1}(x, t_1)\). The iterative process then proceeded until (12) was satisfied. With an appropriate initial guess for the iteration to start with (the initial conditions were usually selected), the iteration converged very quickly.

One should notice that the iteration in (11) is quite efficient in the sense that it is linear and non-transient and most importantly, matrix \(T\) is the only one that needs to be updated after each iteration is completed. With this excellent property, not only can the storage space be reduced (mainly due to the DRM in the Laplace space), but also a large amount of computational time, which would otherwise be required to update other matrices, can be saved.

### 3 Numerical Examples

Unlike the linear cases studied by Zhu et al. [1] and Satravaha et al. [9], in which the numerical solutions can always be compared with analytical solutions, there were no analytical solutions to be compared with when eqn (1) was solved numerically in the nonlinear realm (i.e., \(n \geq 2\)). Therefore, in order to verify the iteration scheme described in the previous section, we chose the function \(g(x)\) to be exactly the same as that used by Zhu et al. [2] so that we were able to compare our results with theirs.

In one of the examples provided by Zhu et al. [2], the heating of a square slab using microwave energy was studied with the assumption that an exponential decay of the electric-field is in the \(x\)-direction only. For that problem, the governing equation (1) is of the form

\[\nabla^2 u = \frac{\partial u}{\partial t} - \beta e^{-\gamma x} u^n, \tag{13}\]

which is subject to boundary and initial conditions

\[u = 1, \quad \text{on } x = 0, 1 \text{ and } y = 0, 1, \tag{14}\]

\[u = 1, \quad \text{at } t = 0. \tag{15}\]
Table 1: Temperature at selected points on the slab for the case $n = 2$: (a) $\beta = 4.7$, $\gamma = 0$ and $t = 1.7$; (b) $\beta = 11$, $\gamma = 2$ and $t = 1.6$; (c) $\beta = 21$, $\gamma = 4$ and $t = 2.1$

<table>
<thead>
<tr>
<th>$x$</th>
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<th>Zhu et al. [2]</th>
<th>LTDRM</th>
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<td>1.496</td>
<td>1.482</td>
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</table>

Table 2: Temperature at selected points on the slab for the case $n = 3$: (a) $\beta = 2.8$, $\gamma = 0$ and $t = 1.9$; (b) $\beta = 6.6$, $\gamma = 2$ and $t = 1.5$; (c) $\beta = 12.3$, $\gamma = 4$ and $t = 1.6$

<table>
<thead>
<tr>
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<tbody>
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<td></td>
<td>$a$</td>
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For the nonlinear cases, Zhu et al. [2] used the coupled DRM and a finite difference scheme in time domain to obtain their numerical solutions when $n = 2,3$. We ran our model with the same power $n$ and compared our results with theirs.

To implement the LTDRM, the boundaries were discretised into 40 constant elements and 36 internal nodes were placed inside of the computational domain. The tolerance for stopping the iteration, $\varepsilon$, was chosen to be 0.001, and the initial conditions were always used for the first iteration. Numerical results of temperature at some selected points on the slab, for the cases when $n = 2$ and $n = 3$, are shown in Tables 1 and 2, respectively. The corresponding values obtained by Zhu et al. [2] for three different values of $\beta$, $\gamma$ and $t$ are also tabulated in Tables 1 and 2 for the purpose of comparison. We have averaged the differences between the two solutions and found that the agreement between the two solutions was remarkable. For weakly nonlinear cases ($n = 2$), the average difference between the two solutions for $\beta = 4.7$ (case (a) in Table 1) is only $5.0 \times 10^{-4}$. For larger $\beta$ values, the average differences slightly increase to $1.67 \times 10^{-3}$ and $8.0 \times 10^{-3}$ for cases (b) and (c), respectively. But they are still very small, indicating an overall
good agreement between the two solutions. When the nonlinearity becomes stronger, the average differences for the cases (a) and (b) in Table 2 just increase slightly ($3.67 \times 10^{-3}$ and $2.17 \times 10^{-3}$, respectively) whereas the average difference for the case (c) becomes even smaller ($3.67 \times 10^{-3}$). These results are very encouraging as they show that not only does the newly proposed iteration scheme work, but also it is quite robust in dealing with the cases with weak as well as strong nonlinearities.

A key achievement in Zhu et al. [2] was to use their DRBEM model to predict the critical $\beta$ value at which the so-called “hot-spots” occur. Hot-spots are the localised areas of high temperature which develop as the material is being irradiated [2]; a correct prediction of “hot-spots” occurrence is important in any industrial process in which a microwave heating process is involved. It is therefore quite interesting to examine whether or not the LTDRM can be used to correctly predict critical $\beta$ values as well. To calculate the critical $\beta$ values, a simple loop was added to our program used to calculate the solution at any observation time. With the steady-state of eqn (13) defined as the difference of two solutions at each collocation points being less than 0.01%, our results for the critical $\beta$ values, $\beta_c$, are tabulated in Table 3 for the cases where $n = 2, 3$. Once again, for the purpose of comparison, the $\beta_c$ values obtained by Zhu et al. [2] and by using the Frank-Kamenetskii’s approximation method [2], [11] are also listed in Table 3. Our results seem to be very close to those obtained by Zhu et al. [2] using their numerical model but slightly larger than those obtained by Zhu et al. [2] using the Frank-Kamenetskii’s approximation method, especially when $n$ is larger. As explained in Zhu et al. [2], the errors involved with using the first eigenfunction to represent the unknown function are expected to become larger when $n$ is larger.

As far as the numerical efficiency is concerned, Zhu et al. [1] and Satravaha et al. [9] demonstrated the excellent efficiency of the LTDRM, especially when the solution at a large observation time is calculated. When nonlinear iterations are involved, the numerical efficiency of the LTDRM is even further enhanced as iterations are only performed for a single time step. In contrast, a number of iterations is needed at each time step in a time-domain method, apart from the fact that solutions at a large number of intermediate time steps are still needed before the desired solution at a certain time is obtained. For example, Zhu et al. [2] had to go through 73 iterations before a convergent solution for the case $\beta = 21$ can be obtained at $t = 2.1$. With a same level of accuracy, we obtained our results with
only 18 iterations which of course should be multiplied by the number of solutions needed in the Laplace space (6 for all examples in this paper), resulting in solving eqn (11) at a total number of 108 times. At the first glance, our scheme does not seem to be economical at all. However, as the observation time is increased, the number of iterations associated with a time domain method will increase at least linearly, whereas the number of iterations required by the LTDRM virtually remains the same.

Furthermore, for the particular example we used in this paper, it was found that when $\beta$ values were in the vicinity of $\beta_c$, the efficiency of Zhu et al. [2] worsened dramatically since the steady-state of eqn (13) was reached very slowly. It is therefore very costly to calculate the critical value $\beta_c$ since an extremely large number of solutions at intermediate time steps need to be calculated, which would be discarded eventually, before the final steady-state solution is obtained. On the other hand, there is no such problem for the LTDRM; the number of iterations involved virtually remains the same, regardless of the size of the observation time level. As far as the number of iterations in the LTDRM is concerned, it depends on $\beta$ values; it may vary from 5 iterations for small $\beta$ values to a maximum of 20 iterations with $\beta$ being very close to $\beta_c$. Clearly, the LTDRM is much more efficient compared to any time-domain scheme if the solution at a large observation time is to be sought.

4 Conclusions

The LTDRM proposed by Zhu et al. [1] has been extended to solve time-dependent diffusion problems with nonlinear source terms. As illustrated by the example given in this paper, the method can be easily extended to other types of nonlinear diffusion problems once the main difficulty is circumvented.

An extension of the LTDRM to nonlinear cases is far from trivial since it is obvious that we cannot perform the Laplace transform on nonlinear terms. However, in this study, we have successfully constructed a nonlinear iteration, by which accurate numerical solutions were obtained. Such a success has shed light on designing nonlinear iteration procedures for other types of diffusion equations.

Once again, a great advantage of the LTDRM in dealing with nonlinear diffusion equations is its numerical efficiency. Although several LTDRM solutions in the Laplace space are required for a reliable inverse Laplace transform, the overall computational efficiency is still much higher compared to any time-domain method, especially when the solution at a large observation time is needed. The fact that only a single time step is needed in the LTDRM shows that not only does a highly computational efficiency in terms of computer storage and computational time has been achieved, but also the numerical accuracy is enhanced in the sense that less accumulated errors are involved in a calculation compared to time-domain methods.
ACKNOWLEDGEMENTS

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