‘E’ can mean eigenfunction in BEM

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INTRODUCTION:

An examination of the history of what we now call boundary element methods (BEM) would quickly reveal that this was not the original name for this method. The key idea then had been the reduction of dimensionality by taking a domain problem to a boundary problem. Emphasis was on the formulation, i.e. the boundary integral equation method (BIEM), e.g. Cruse and Rizzo. This formulation, then also called the surface integral equation, was equivalent to any partial differential equation formulation. It is only in the approximation of this integral equation that errors, other than those inherent in any mathematical modeling of an engineering problem, are introduced.

What is of particular interest here is the fact that "other than element" solution methods have a long history within this field. Some of these have been "dead-ends" in the sense that they did not seem practical while others have found wide application in certain fields, essentially those not dominated by finite element methods. The widest area of application of these methods was to wave scattering and radiation problems in infinite domains, e.g. underwater acoustics. The restrictive use of the "element" concept in BEM has led much of the thinking in BEM to be modeled after FEM even though the basic formulations are quite distinct. For example, the use of a "constant shape function" in BEM leads to a discontinuous "stepped" representation of the results. Treating the dependent variables in a
weighted mean value sense, more appropriate to integrals, leads to a more physically appealing result, e.g. Shaw\textsuperscript{2}, with the same effort. While this caveat is related to element methods, it indicates that there may be value in reexamining the foundations of current BEM.

**ALTERNATIVE SOLUTION METHOD DESCRIPTIONS:**

There are many choices in the way a boundary integral may be solved. One involves the relative location of the field or solution points and the source or integration points. These are both placed on the actual physical boundary in BEM but this is rarely the case for "non-element" methods. Consider some possible approaches.

A: "Classical" Methods: These include

a) Separation of Variables - when this works, it is actually a boundary method in that the entire solution is obtained as an eigenfunction expansion whose coefficients are obtained directly in terms of the given boundary conditions,

b) Iteration - this is one of the classical methods for the solution of integral equations. Unfortunately, the singular integral equations found in BIEM appear to lead to convergence difficulties.

c) Trefftz Methods - these are the counterpart of separation of variable methods when there is no set of basis functions available for the geometry under consideration. They use any convenient set of basis functions and lead to systems of coupled algebraic equations for the series coefficients. This should sound familiar to the BIEM/BEM community in that we use "A" convenient Green's function rather than "THE" appropriate one for a given geometry and boundary condition, e.g. Herrera\textsuperscript{3}.

d) Asymptotic Expansions - these involve solution of the original BIE for very large or very small values of the parameters of the problem. These could be geometrical, e.g. slight changes from a geometry whose solution is already known, or physical, e.g. low frequencies in time harmonic wave problems, e.g. Shaw\textsuperscript{4} for the geometrical case.

B: "Modern Approximate" Methods: These move the field and source points to other locations than those used in BEM and are not
really modern at all. They arose primarily in wave type problems where the domains under consideration are often infinite and a reduction to a finite boundary surface is almost irresistible and form a significant part of the early (pre 1970) applications of the BEM.

a) T Matrix Methods- this method is still actively used in the wave scattering and radiation community whether the waves be acoustic, elastic or electromagnetic. The actual boundary is considered as an equivalent surface of sources and doublets and is the surface on which integrations are performed, but the solution points are placed on two auxiliary surfaces, one outside of the original domain and one inside of it. The variables are then expanded into series of in terms of some convenient basis set of eigenfunctions which are not necessarily those for the original geometry. This method has been applied primarily to wave scattering problems in infinite domains, e.g. Waterman for the original paper and Varadan and Varadan for a review,

b) Null Field Methods- this approach places the solution points outside of the original domain, where the BIE would give a null field. This may be solved with either element or eigenfunction expansion methods, e.g. Bates and Wall. In the latter case, it is equivalent to using a T Matrix Method but stopping part-way. The approach has been used in its element form in conjunction with standard BEM techniques to develop overdetermined systems of equations for the resolution of specific difficulties such as the fictitious interior eigenvalue problem, e.g. Schenck.

c) Embedding Integral Methods- here the surface of sources, on which integrations are carried out, is chosen for convenience and is placed outside of the original domain with its strengths adjusted to match the given boundary conditions on the original boundary. This formulation may also be solved by either element or eigenfunction expansion methods. This element approach is actually one of the earliest applications of integral equation methods, e.g. Munk and Von Karman. The eigenfunction expansion approach is equivalent to the Trefftz method, e.g. Shaw and Huang and has been applied to non-wave problems as well, e.g. Shaw, Huang and Zhao, with partitioning.
DETAILS OF SOME EIGENFUNCTION EXPANSION METHODS:

The T Matrix Method and the Embedding Integral Method will be discussed here briefly; the other methods are left to the references.

a) The T Matrix Method (including the null-field method):
An exterior time harmonic wave scattering problem is considered. The governing integral equation for an acoustic velocity potential, \( \phi(\vec{r}) \), in an infinite domain \( \mathbb{D} \) in which a finite body with a boundary \( \mathbb{B} \) is immersed, with \( Q(\vec{r}) = 0 \) but with an incident wave field, \( \phi_\omega(\vec{r}) \), is

\[
\begin{align*}
&c \phi(\vec{r}_0) = \phi_\omega(\vec{r}_0) + \int_{\mathbb{B}} \left[ G(\vec{r}, \vec{r}_0) \frac{\partial \phi(\vec{r})}{\partial n} - \phi(\vec{r}) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} \right] dS(\vec{r}) \\
&\text{where } c = 0 \text{ for } \vec{r}_0 \text{ outside of } \mathbb{D}, 1 \text{ for } \vec{r}_0 \text{ within } \mathbb{D} \text{ and } 1/2 \text{ for } \vec{r}_0 \text{ at a smooth boundary point. } G(\vec{r}, \vec{r}_0) \text{ is well known with expansions in a variety of basis sets, e.g. Morse and Feshbach}^{13}. \text{ Use a set of basis functions } \Psi_p(\vec{r}) \text{ and write the expansions}
\end{align*}
\]

\[
\begin{align*}
\phi_\omega(\vec{r}) &= \sum_{p=0}^{\infty} a_p \text{ Re}\left[ \Psi_p(\vec{r}) \right] ; \quad \phi_\beta(\vec{r}) = \phi(\vec{r}) - \phi_\omega(\vec{r}) = \sum_{n=0}^{\infty} f_n \Psi_n(\vec{r}) \\
G(\vec{r}, \vec{r}_0) &= \beta \sum_{n=0}^{\infty} \left\{ \begin{array}{ll}
\Psi_n(\vec{r}_0) \text{ Re}\left[ \Psi_n(\vec{r}_0) \right] ; & r > r_0 \\
\Psi_n(\vec{r}_0) \text{ Re}\left[ \Psi_n(\vec{r}) \right] ; & r_0 > r
\end{array} \right.
\end{align*}
\]

The real part of \( \Psi_n(\vec{r}) \) is used in the incident field expansion to have a finite wave potential at the origin and the complex form of \( \Psi_n(\vec{r}) \) is required for the scattered wave potential in order to have outgoing waves from the obstacle. The factor \( \beta \) is typical of the T matrix notation. The governing integral equation is then written for \( \vec{r}_0 \) on the circumscribed surface, \( r_0 > r \), and for \( \vec{r}_0 \) on the inscribed surface, \( r_0 < r \). For a rigid scatterer, with \( \partial \phi/\partial n \) equal to zero on the boundary, this gives

\[
\begin{align*}
\phi_\phi(\vec{r}_0) &= \sum_{n=0}^{\infty} f_n \Psi_n(\vec{r}_0) = \beta \int_{\mathbb{B}} \phi(\vec{r}) \sum_{n=0}^{\infty} \Psi_n(\vec{r}_0) \text{ Re}\left[ \frac{\partial \Psi_n(\vec{r})}{\partial n} \right] dS(\vec{r}) \\
\phi_\omega(\vec{r}_0) &= \sum_{p=0}^{\infty} a_p \text{ Re}\left[ \Psi_p(\vec{r}_0) \right] = \beta \int_{\mathbb{B}} \phi(\vec{r}) \sum_{p=0}^{\infty} \text{ Re}\left[ \Psi_p(\vec{r}_0) \right] dS(\vec{r})
\end{align*}
\]
requiring, through the independence of the set of basis functions,

\[ f_n = - \beta \int_{\mathcal{B}} \phi(\hat{r}) \text{Re}[\partial \Psi_n(\hat{r})/\partial n] \, dS(\hat{r}); \quad a_p = \beta \int_{\mathcal{B}} \phi(\hat{r}) \partial \psi_P(\hat{r})/\partial n \, dS(\hat{r}) \]  \[12\]

The field on the scattering surface, \( \phi(\hat{r}) \), is expanded in an appropriate set of basis functions, \( Y_m(\hat{r}_0) \), related to \( \Psi_m \) or its derivatives,

\[ \phi(\hat{r}) = \sum_{m=-\infty}^{\infty} \alpha_m \, Y_m(\hat{r}) \]  \[13\]

Then \( f_n \) and \( a_p \) are related through the elimination of \( \alpha_m \). For real \( Y_m \), the definition of the \( Q \) matrix is

\[ Q_{mn} = \int_{\mathcal{B}} Y_m(\hat{r}) \, \partial \Psi_n(\hat{r})/\partial n \, dS(\hat{r}) \]  \[14\]

leading to the equations

\[ f_n = - \beta \sum_{m=-\infty}^{\infty} \alpha_m \text{Re}[Q_{mn}]; \quad a_p = \beta \sum_{m=-\infty}^{\infty} \alpha_m \, Q_{mp} \]  \[16\]

The second part of eq. \[16\], relates the coefficients \( \alpha_m \) to the coefficients \( a_p \), i.e. a null-field solution. However, the cost of going an extra step and including the first part is low since \( Q_{mn} \) has already been calculated, and appears to improve the solution. Then

\[ f_n = - \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} a_p \, Q_{mp}^{-1} \text{Re}[Q_{mn}] = \sum_{p=0}^{\infty} a_p \, T_{pn} \]  \[17\]

If the set of basis functions \( Y_m \) is complex, these are modified somewhat in detail, although the \( Y_m \) are usually taken to be \( \text{Re}[\Psi_m(\hat{r})] \). The use of a real basis set \( Y_m \) allows for the "fictitious interior eigenvalue" difficulty to arise in the \( T \) Matrix method, e.g. Waterman\(^5\). Using the full complex set for \( Y_m \) avoids this, Martin\(^{14}\).

b) The Embedding Integral Method: This is one of the oldest and most physical of the boundary methods. Here the original geometry is embedded in a larger domain with a simpler, e.g. separable, shape. The boundary conditions on this "embedding" surface may related to the given boundary conditions on the original "embedded"
Take, as a 2D example, an embedding circle, $S_e$, inscribed within the original geometry boundary, $\mathcal{B}$, for an exterior problem, i.e. outside of the original domain, $\mathcal{D}$. If this embedding surface is taken as a surface of sources of strength $\sigma$, it produces a solution

$$\phi(\mathbf{r}) = \int_{S_e} \sigma(\mathbf{r}_e) \ G(\mathbf{r}, \mathbf{r}_e) dS_e$$

with a derivative, proportional to the flux,

$$\frac{\partial \phi(\mathbf{r})}{\partial n} = \int_{S_e} \sigma(\mathbf{r}_e) \ \partial G(\mathbf{r}, \mathbf{r}_e)/\partial n \ dS_e$$

Then expand, using the previous form for $G$ with $r_0 > r$,

$$\sigma(\mathbf{r}_e) = \sum_{m=0}^{\infty} \alpha_m \ \Psi_m(\mathbf{r}_e); \ \Phi(\mathbf{r}_0) = \sum_{m=0}^{\infty} B_m \Psi_m(\mathbf{r}_0) \text{ on } r_e = b$$

The boundary conditions determine $\alpha_m$ which then determines $B_m$. The embedding surface radius $r_e$ does not appear explicitly in the solution. This procedure has been used in Helmholtz problems, e.g. Shaw, Huang and Zhao\textsuperscript{12}; full details are given in Huang\textsuperscript{15}.

**CONCLUSIONS:**

The eigenfunction expansion methods may be summarized as having a: the field points lie inside of the obstacle (outside of the original domain, $\mathcal{D}$) while the integration takes place over the obstacle surface, (the "null-field" approach), b: half of the field points lie inside the domain $\mathcal{D}$ and half lie outside, with integrations on the obstacle, (the "T Matrix" method), c: the field points lie on the obstacle surface, but the integration is taken over some other known surface, $S_e$, outside of $\mathcal{D}$, (the "Embedding Integral Equation Method"). They are of use in the approximate solution of engineering problems, but are subject to some drawbacks. The T Matrix method has been applied almost exclusively to non-mixed boundary condition problems, i.e. where the given boundary conditions are all of the same type. When mixed boundary conditions are used, there may be some
difficulty in calculating the expansion coefficients $a_p$. The Embedding Integral Equation method does not suffer from this difficulty since the expansion and integration are not on the original boundary and solutions are usually found by collocation with boundary conditions.

APPENDIX:

Some results are given for the two dimensional example of a plane wave, $\exp(ikx)$, scattered by a rigid circle of radius $a$.

a: Classical Separation of Variables.

$$\Phi(\vec{r}) = \sum_{m=0}^{\infty} \epsilon_m i^m [J_m(kr) - (J_m'(ka)/H_m^{(1)')(ka)) H_m^{(1)}(kr)] \cos(m \theta) \quad [A-1]$$

where $\epsilon_m$ is the Neumann function and is 1 for $m = 0$ and 2 otherwise.

b: "T Matrix Method". In this case the potential is calculated at $r_0 = a^-$ and at $r_0 = a^+$. With the notation as above, this results in

$$f_m = -(J_m'(ka) J_m(ka)/[H_m^{(1)')(ka) J_m(ka)]) a_m \quad (\text{no sum on } m) \quad [A-2]$$

The equation is indeterminate at $J_m(ka) = 0$ although the correct limit is found as $k$ approaches these interior eigenvalues.

d: "Embedded Field Method". A surface of simple sources of strength $\alpha$ is placed on the circular 'boundary', $r = b$, where $b < a$ and the rigid boundary condition at $r = a$ is enforced. This leads to coefficients for the potential on the boundary as

$$B_m = \epsilon_m i^m [J_m(ka) - (J_m'(ka) J_m(kb)/[H_m^{(1)')(ka) J_m(kb)]) H_m^{(1)}(ka)] \quad [A-3]$$

which is the correct solution at all points except for the interior eigenvalues of the artificial surface, $r = b$, when $J_m(kb) = 0$, where this is the correct solution in the limit. The difficulty then has been shifted from the eigenvalues of the original obstacle and boundary conditions to those of the artificial surface.
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