The hybrid boundary element method for the analysis of solids
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INTRODUCTION

The hybrid boundary element method, as developed in the Civil Engineering Department at PUC/RJ, may already be considered a well-established formulation for problems of elasticity and potential. Among other publications, several articles have been presented at BEM International Conferences, since 1987, dealing with the basic theory \cite{1}, body forces \cite{2}, special applications \cite{3, 4} and transient problems \cite{5}.

The present paper describes the implementation of a three-dimensional analysis program \cite{6}. In a first step, the basic equations are introduced and the most relevant numerical aspects are discussed. Then follows a general outline of the program, as regards stress analysis and post-processing of results. Some examples are displayed for illustration of the capabilities of the program, mainly concerning the ease of data-handling, a characteristic of the hybrid variational formulation implemented.

BASIC EQUATIONS

The basic equations of the hybrid boundary element method are

\begin{equation}
F \ p^* = H \ d + b,
\end{equation}

a nodal displacements compatibility equation, and

\begin{equation}
H^T p^* = p - t,
\end{equation}

a nodal forces equilibrium equation. In these equations, \(d \equiv d_m\) are nodal displacement parameters, used for the description of the displacements \(u_i\) along the boundary, in terms of an interpolation function \(u_{im}\):

\begin{equation}
u_i = u_{im} d_m \text{ on } \Gamma,
\end{equation}

in such a way that prescribed boundary conditions

\begin{equation}
\bar{u}_i = u_{im} \bar{d}_m \text{ on } \Gamma_u
\end{equation}
are satisfied.

The stresses in the domain are represented by Kelvin's fundamental solution, in terms of which one defines the flexibility matrix $F = F_{mn}$ of Eq. (1):

$$F_{mn} = \int_{\Gamma} p^* u^*_{in} \, d\Gamma + \int_{\Omega} \Delta_{im} u^*_{in} \, d\Omega.$$  \hspace{1cm} (5)

In this equation, $p^*$ relates tractions $T^*_i$ along the boundary to singular forces $p^* = p^*_m$ of Kelvin's fundamental solution:

$$T^*_i = \sigma^*_{ijm} \eta^*_j p^*_m = p^*_m p^*_m$$  \hspace{1cm} (6)

and $u^*$ transforms these same singular forces $p^*_m$ into displacements:

$$u^*_i = u^*_i p^*_n.$$  \hspace{1cm} (7)

Moreover, $\Delta_{im}$ is a Dirac-function that expresses the singularity of $p^*_m$.

The kinematic matrix $H = H_{mn}$, which appears in Eqs. (1) and (2), is expressed by

$$H_{mn} = \int_{\Gamma} p^* u^*_{in} \, d\Gamma + \int_{\Omega} \Delta_{im} u^*_{in} \, d\Omega.$$  \hspace{1cm} (8)

Besides that, there is in Eq. (1) a vector $b = b^*_m$ of equivalent nodal displacements:

$$b^*_m = \int_{\Gamma} p^* u^*_{in} \, d\Gamma + \int_{\Omega} \Delta_{im} u^*_{in} \, d\Omega.$$  \hspace{1cm} (9)

related to the displacements $u^*_i$ that one may obtain as a particular solution of the differential equilibrium equation

$$\sigma^*_{ij} + F^*_i = 0 \quad \text{in } \Omega,$$  \hspace{1cm} (10)

for prescribed body forces $F^*_i$.

There are in Eq. (2) two vectors of equivalent nodal forces:
Boundary Elements

\[ t \equiv t_n = \int_\Gamma \sigma_{ij}^p \eta_j u_i \, d\Gamma, \quad (11) \]

in which \( \sigma_{ij}^p \) is a particular solution of Eq. (10), and

\[ p \equiv p_n = \int_\Gamma T_i u_i \, d\Gamma, \quad (12) \]

for prescribed tractions \( T_i \) along the boundary \( \Gamma \) (some elements of \( p \) may turn out to be actually reaction forces, at part \( \Gamma_u \) of \( \Gamma \) where the displacements \( \bar{u}_i \) are known [1]).

Substitution of \( p^* \) from Eq. (1) into Eq. (2) yields:

\[ K d = p - t - H^T F^{-1} b, \quad (13) \]

in which

\[ K = H^T F^{-1} H \quad (14) \]

is a symmetric, positive semi-definite matrix. The formal presentation of the equations above must be complemented by one important remark. The elements about the main diagonal of matrix \( F \), for the degrees of freedom referred to the same nodal point, cannot be evaluated in terms of the integration indicated in Eq. (5), due to a strong singularity of the type \( 1/r^2 \). On the other hand, both Eqs. (1) and (2) indicate transformations that are orthogonal to rigid body displacements and to non-equilibrated forces, respectively. Then, if one defines \( W = W_m^s \) as a six-columns orthogonal basis of rigid body displacements, it follows that

\[ H W = 0, \quad (15) \]

a well known property. As a consequence, there is another orthogonal basis \( V \), such that

\[ H^T V = 0. \quad (16) \]

It follows, then, that a sufficient condition for determining the elements about the main diagonal of \( F \) is that also

\[ F V = 0. \quad (17) \]

For potential problems, there is one degree of freedom for each discretized node and Eq. (17) results in a series of single uncoupled equations for each one of the diagonal unknowns. In case of three-dimensional elasticity, however, Eq. (17) implies, for each node, in a \( 6 \times 3 \) equations system with \( 3 \times 3 \) unknowns, or, if one takes into account that \( F \) must be actually symmetric, a \( 6 \times 3 \) equations system with 6
unknowns. This equation system may be solved in terms of least squares and
turns out to be quasi-consistent. Its consistency increases with
increasing number of nodal points [6].

As a consequence of Eq. (17), the inversion of \( F \) indicated in Eqs. (13) and (14) must be interpreted in terms of generalized inverses [7]. It
turns out that the product \( H^T F^{-1} \) is unique and may be expressed as

\[
H^T F^{-1} = H^T (F + V V^T)^{-1},
\]

in which \( F + V V^T \) is positive definite.

After solution of Eqs. (1) and (2), the stresses at any point inside
the domain may be obtained from the singular forces \( p^* \):

\[
\sigma_{ij} = \sigma_{ij}^* p_{im}^* + \sigma_{ij}^p.
\]

The corresponding displacements may also be evaluated as

\[
u_i = \tilde{u}_{im}^* p_{im}^* + \tilde{u}_{ij}^p + u_{ij}^r \ W_{ms} \ d_m,
\]

in which \( \tilde{u}_{im}^* \) and \( \tilde{u}_{ij}^p \) are functions obtained by orthogonalizing \( u_{im}^* \) and \( u_{ij}^p \)
with respect to rigid body displacement functions \( u_{is} \).

For simply constrained structures, that is, when the vector of nodal
forces \( p \) is completely known (Neumann boundary conditions), the
formulation introduced above simplifies drastically, since Eq. (2) alone
is sufficient for the determination of the stresses in the body, as a
function of the singular forces \( p^* \). Since Eq. (2) is a singular system, it
must be complemented by the condition that admissible solutions of \( p^* \) are
orthogonal to \( V \). Then, the equation system

\[
H^T p^* = p - t
\]

\[
V^T p^* = 0,
\]

although rectangular, is consistent and easy to solve [1, 6].

After solution of Eqs. (21), stresses and displacements inside the
body may be obtained by Eqs. (19) and (20), respectively, although the
contribution of the rigid body displacements shall remain unknown, since
the nodal displacements \( d \) are not determined by Eqs. (21).

**NUMERICAL EVALUATION OF THE MATRICES**

The equations introduced above were implemented in a general analysis
program using isoparametric triangular and quadrilateral elements with
linear or quadratic interpolation functions. The same interpolation func-
tions were used to describe distributed forces over the boundary. Several
analytical expressions of body forces were also included in the code.

Three general cases of numeric integration have to be considered, for
the evaluation of the integrals introduced above. Since the integration has to be carried out element by element, the first case corresponds to regular integrands, as for Eqs. (11) and (12), over the whole boundary, or for Eqs. (5), (8) and (9), when the integrand refers to singular forces \( \mathbf{p}_m \) applied at points outside the element.

On the other hand, when the displacement function \( \mathbf{u}^* \) in Eq. (5) refers to forces \( \mathbf{p}_m^* \) applied at points of the boundary element, a \( 1/r \) singularity appears, resulting in an improper integral, although defined in the Riemannian sense. In this case, a simple transformation to polar coordinates, as illustrated in Fig. 1, eliminates the singularity.

![Figure 1](image-url)  
Figure 1. Polar coordinate system for a boundary element subdivided into triangles.

The third integration case occurs when the traction function \( \mathbf{p}_{im}^* \) in Eqs. (5), (8) and (9) refers to forces \( \mathbf{p}_m^* \) applied at points of the boundary element, giving rise to a \( 1/r^2 \) singularity. The corresponding singular integral has to be dealt with in two steps. Firstly, the integral is evaluated in the sense of a Cauchy principal value, as regards the whole integration boundary, which means a finite-part integration over each element affected by the singularity. A complementary, singular-part integral has to be evaluated in a second step, as shall be outlined in the next section. For carrying out the finite-part integration over the element in question, a transformation to polar coordinates is performed, as described in the previous case, according to Fig. 1. As a consequence, the integral is regular in terms of the variable \( \theta \), but still has a \( 1/p \) singularity, to be handled in terms of finite part.

The authors are not intended to write a review of integration procedures, as regards three-dimensional problems. Many researchers have contributed in the last years [8-12], as reported in [6], and the treatment outlined above and implemented in [6] does not differ in nature from the best procedures recommended in the technical literature for dealing with the problem, although it has been developed independently.

But there is a remarkable difference in the way the strong singularity \( 1/p \) is dealt with in terms of transformation to polar coordinates. Firstly, instead of dealing with concepts such as the ones
related to a tangent plane drawn at the singularity point [11, 12], the authors simply apply the correct procedure to normalize an integration interval, expressed in general as:

\[ \int_{\Gamma} \frac{1}{r} f(r) \, d\Gamma = \int_{0}^{1} \frac{1}{\rho} \left( \frac{\rho}{r} f |J| \right) \, d\rho + f(r) \ln \left( \frac{dr}{d\rho} \right) \bigg|_{\rho=0}. \tag{22} \]

In equation above, the integration is interpreted in terms of finite part, over an one-dimensional curved segment of boundary \( \Gamma \). The function \( f(r) \) is the regular part of the integrand and \( |J| \) is the Jacobian of the coordinates transformation from \( r \) to \( \rho \). The last term in Eq. (22) was obtained for the first time in the frame of the present research work, to the authors' best knowledge [13]. Other researchers had used Eq. (22) with \( r_{\text{max}} \) – the maximum value of the radius \( r \) in the integration interval – as the argument of the logarithmic function in Eq. (22), which is only correct for straight boundaries. The adequacy of Eq. (22) is demonstrated in [13].

Another, to a certain extent innovative, procedure related to the present research work [13, 14] was the use of the transformation

\[ \int_{0}^{1} \frac{1}{\rho} \left( \frac{\rho}{r} f |J| \right) \, d\rho = \int_{0}^{1} \frac{1}{\rho} \left( \frac{\rho}{r} f |J| - f(0) \right) \, d\rho, \tag{23} \]

which enables carrying out a Gauss-Legendre quadrature of the regular integral at the right-hand side of equation above, instead of the cumbersome, less accurate scheme suggested by Kutt [15].

The integration of matrix \( F \), as given in Eq. (5), becomes very involved, when both functions \( p_{im}^* \) and \( u_{in}^* \) are singular inside the same element. The solution was to subdivide the element into more triangles than indicated in Fig. 1, in order to have one single integration case, as outlined above, for each triangle. The authors are investigating an alternative procedure, based on the achievements of papers [14] and [16], that should eliminate the necessity of polar transformations, at least for plane elements.

**EVALUATION OF THE SINGULAR PART OF MATRIX H**

When the indices \( m \) and \( n \) of the kinematic matrix \( H \) refer to degrees of freedom of the same nodal point, the corresponding submatrix \( H_{ij} \) comprises both a finite and a singular part. The evaluation of the finite part was discussed in previous section. The singular part is usually called \( C_{ij} \) in the literature and has the general expression:

\[ C_{ij} = \int_{\Gamma} p_{ij}^* \, d\Gamma + \delta_{ij}. \tag{24} \]
The expression of $C_{ij}$ is given by Hartmann \[17\] for a few particular cases in which the singularity pole is the vertex defined by orthogonal planes. But, in the general case of a vertex formed by arbitrarily shaped boundary elements, there is no close expression available (a presumably close expression was shown in an oral presentation at the BEM 14, held in Seville, but the authors had no access to the written material; moreover, the present research work \[6\] was already concluded at that time).

Figure 2a shows, as a matter of illustration, a spheric surface segment $\Gamma_0$ — over which one intends to evaluate the integral of Eq. (24) — that surrounds a vertex formed by four arbitrarily disposed boundary elements. If one defines a local coordinate system $(u, v, n)$, the planes formed by the axis $n$ and the tangents at the vertex of the intersection of two adjacent elements will cut the spheric surface $\Gamma_0$ into as many sectors as the number $e$ of elements converging at the vertex. Then

$$\int_{\Gamma_0} \mathbf{p}_{ij}^* \, d\Gamma = \sum_{s=1}^{e} \int_{\Gamma_s} \mathbf{p}_{ij}^* \, d\Gamma. \quad (25)$$

Figure 2. a) Spheric surface of the boundary $\Gamma_0$; b) Example of vectors $\mathbf{p}$ and $\mathbf{q}$ tangent to an element at the singularity pole.

An adequate way of defining the local coordinate system $(u, v, n)$ is described in the following.

The unity vector $\mathbf{n}$ may be defined as the average of all outward normals $\mathbf{n}_s$ of the boundary elements, calculated at the common vertex:

$$\mathbf{n} = \frac{\sum_{s=1}^{e} \mathbf{n}_s}{\sqrt{\sum_{s=1}^{e} \mathbf{n}_s \cdot \mathbf{n}_s}}. \quad (26)$$

Equation above guarantees that $\mathbf{n}$ also points outward. Now, consider two vectors $\mathbf{p}$ and $\mathbf{q}$ tangent to the edges of a given boundary element at the singularity pole (Fig. 2b):

$$\mathbf{p} = \frac{\partial \mathbf{x}_i}{\partial \xi} |_{\xi_0} \mathbf{e}_i \quad \text{and} \quad \mathbf{q} = \frac{\partial \mathbf{x}_i}{\partial \zeta} |_{\zeta_0} \mathbf{e}_i, \quad (27)$$
where \( \mathbf{e}_i \) are the unity vectors corresponding to the global coordinates \( x_i \equiv (x, y, z) \) and \( (\xi, \zeta) \) is the natural coordinate system of the element. Then, the second coordinate direction of the system \((u, v, n)\) may be obtained as the vectorial product:

\[
\mathbf{v} = \mathbf{n} \times \mathbf{p}
\]

and, consequently,

\[
\mathbf{u} = \mathbf{v} \times \mathbf{n}.
\]

As a result, if one introduces the spheric coordinate system of Fig. 3, Eq. (25) may be expressed as

\[
\int_{\Gamma} p_{ij}^* \, d\Gamma = \sum_{s=1}^{e} \int_{0}^{\theta_f} \int_{0}^{\varphi_f} f_{ij}(\theta, \varphi) \sin \varphi \, d\varphi \, d\theta,
\]

in which

\[
f_{ij}(\theta, \varphi) = \frac{-1}{8\pi(1-v)} \left[ (1 - 2v) \delta_{ij} + 3 \frac{r_i}{r_j} \right]
\]

is a regular function, since the definition of a boundary segment in spheric coordinates

\[
d\Gamma = r^2 \sin \varphi \, d\varphi \, d\theta
\]

eliminates the initial singularity \(1/r^2\).

Figure 3. Spheric coordinate system and integration limits.

The matrix

\[
\mathbf{T} = \begin{bmatrix}
u_x & u_y & u_z \\
v_x & v_y & v_z \\
n_x & n_y & n_z \\
\end{bmatrix}
\]

transforms global coordinates \((x, y, z)\) into local coordinates \((u, v, n)\).
These coordinates may also be given in terms of spheric coordinates as

\[ <u, v, n> = r \phi \equiv r <\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi>. \]  

(34)

If one expresses the derivatives of \( r \), given in Eq. (31), in terms of spheric coordinates, using matrices \( T \) and \( \phi \) defined above, Eq. (30) becomes:

\[
\int_{\Gamma_0} p_{ij}^* d\Gamma = \frac{-1}{8\pi(1-v)} \sum_{s=1}^{\text{c}} \left\{ \begin{array}{c} T^T \int_{0}^{\theta_f} \int_{0}^{\phi_f} \phi^T \sin\phi \phi^T d\phi d\theta T + \\
+ (1 - 2v) \delta_{ij} \int_{0}^{\theta_f} \int_{0}^{\phi_f} \sin\phi d\phi d\theta \end{array} \right\}.
\]  

(35)

The limit of integration \( \theta_f \) of one element \( s \) is

\[ \theta_f = \arctan(q_v / q_u), \]  

(36)

as one may infer from Fig. 3. The limit of the coordinate \( \phi \) is given, for element \( s \), by the intersection of the spheric surface and the plane that is tangent to the element at the singularity pole:

\[ \eta_u u + \eta_v v + \eta_n n = 0, \]  

(37)

from which follows, according to the definition of \( (u, v, n) \) given in Eq. (34):

\[ \phi_f = \arctan\left( \frac{-\eta_n}{\eta_u \cos\theta + \eta_v \sin\theta} \right). \]  

(38)

The integrands indicated in Eq. (35) are very simple, in principle. Indeed, analytical integration may be carried out easily, in terms of the variable \( \phi \). But, when the integration limit \( \phi_f \) is substituted by its value given in Eq. (38), the first integrand at the right-hand side of Eq. (35) becomes a very complicated function of the variable \( \theta \). Nevertheless, all therms in the integrand are regular, which enables carrying out a Gauss-Legendre quadrature with just a few integration points [6].

The technique outlined above deals with the singular part of the diagonal terms of matrix \( H \), Eq. (8). The \( 1/r^2 \) singularities of matrices \( F \), Eq. (5), and \( b \), Eq. (9), are resolved by means of the same matrix \( C_{ij} \) defined in Eq. (24) for matrix \( H \).

**PROGRAM OUTLINE**

The present three-dimensional formulation was implemented in language C.
All algorithms were written according to concepts of object oriented programming [18], in order to be independent of the type of boundary element considered.

A very simple, interactive graphic three-dimensional boundary element mesh generator was developed. It was based on the definition of isoparametric superelements for the description of the boundary geometry.

The post-processor was based on the graphic system GKS/PUC, with graphic interface IntGraph, as developed by the work group TecGraf at PUC/RJ [6]. A description of this post-processor would be too extensive. A general outline of the options available is given in Fig. 4. The data structure is extremely simple and the graphic representation of results is accomplished immediately, since no integrations need to be performed for the evaluation of displacements and stresses – Eqs. (19) and (20).

![Figure 4. Screen with an outline of the post-processing options.](image)

**NUMERICAL EXAMPLES**

Figure 5a represents a beam submitted to pure bending. Fourteen quadratic elements, comprising 57 nodal points, were used for the discretization of one eighth of the structure, as shown in Fig. 5b. Some results are presented in Tables 1a and 1b.

Figure 6a represents a spheric cavity in an infinite domain submitted to a uniform vertical pressure, considered as a body force. One eighth of the cavity boundary was discretized with 81 linear triangular elements and 55 nodes, according to Fig. 6b. Some results are presented in Table 2, as
compared with values given in [19], for Poisson's ratio \( v = 0.2 \) and vertical pressure \( p = 1 \). Figure 7 illustrates the possibilities of stress representation over some arbitrary cuts in the structure. This representation is very interesting and versatile, since the analyst may retrieve any previous results using the undo option. The results over a plane are calculated and displayed in the screen almost instantaneously, since no integrations are needed for the evaluation of stresses and displacements at internal points.

Figure 5. a) Beam under pure bending \((E = 2.4 \times 10^5, v = 0.2, p_{\text{max}} = 30.)\); b) corresponding boundary discretization.

<table>
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<th>y-axis</th>
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<th>numerical results</th>
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</tr>
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<td>22.5</td>
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Table 1. a) Normal stresses \( \sigma_{yy} \) along y-axis; b) displacements \( u_y \) along x-axis (times \( 10^3 \))

<table>
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</table>

Figure 6. a) Spheric cavity in an infinite domain; b) discretization.
Table 2. Stresses at internal points along the $x$-axis

<table>
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<th>analytical solution</th>
<th>numerical results</th>
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</table>

Figure 7. Representation of $\sigma_{xy}$ stresses over successives cuts.

CONCLUSION

This paper outlined the main features of a program developed for the analysis of solids, in the frame of the hybrid boundary element method. The post-processing options, which take advantage of the formulation, are numerous and probably deserve being dealt with in a separate paper.

But the most important contribution of the research work reported here [6] is related to the numerical evaluation of the singular integrals that appear in the hybrid boundary element method (and in other boundary formulations, too). The technique of choosing a local coordinate system $(u, v, n)$ for dealing with the singular part of an integral with singularity pole $1/r^2$ turned out to be very effective, for general curved boundary elements, as an alternative to the use of rigid body displacements, which is not always feasible.
The evaluation of the flexibility matrix $F$, Eq. (5), is very time consuming, since each term of this matrix demands integration over the whole boundary. But, considering that the expression of $F$ only depends on the boundary geometry, implementation of the main achievements of papers [14] and [16] may contribute to the drastic decrease of computational effort.

The analysis of simply constrained structures (for which only Neumann conditions may be prescribed) requires the evaluation of matrix $H$, only, and is extremely simple, in the frame of the present formulation, particularly if one takes advantage of eventual symmetry or anti-symmetry.

REFERENCES


