Transactions on Modelling and Simulation vol 1, © 1993 WIT Press, www.witpress.com, ISSN 1743-355X

Hypersingular integrals at a corner

L.J. Gray, L.L. Manne
Mathematical Sciences Section, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA

ABSTRACT

For a smooth boundary, hypersingular integrals can be defined as a limit from the interior, the approach direction being taken, for convenience, normal to the surface. A limit definition is also valid at a boundary corner, provided a consistent approach direction is employed for all singular integrals. The analytical formulas for these integrals are more complicated than for a smooth surface, but are easily obtained using symbolic computation. These techniques will be employed to accurately solve the 'L-shaped domain' potential problem considered in the text by Jaswon and Symm. As demonstrated therein, standard boundary element methods applied to this problem yield inaccurate solutions at the corners.

INTRODUCTION

For many applications, a boundary integral analysis can be considerably improved, from the standpoint of simplicity and accuracy of the solution, by employing hypersingular equations (Krishnasamy et al.,[1] Morino and Piva[2]). One specific application is for solving problems in which there is more than one unknown at a boundary corner point (Gray and Lutz[3]). In this case, an equal number of unknowns and equations is achieved by using the hypersingular equation at the corner. While this problem obviously requires the evaluation of corner hypersingular integrals, this situation also arises in various other applications (e.g., treatment of surface cracks).

The simplest example of a hypersingular equation arises in conjunction with

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the two-dimensional Laplace equation $\nabla^2\phi = 0$. As is well known (Brebbia et al.[4]), the corresponding boundary integral form is given by

$$\phi(P) + \int_\Gamma \phi(Q)\frac{\partial}{\partial n}G(P,Q)\,d\Gamma(Q) = \int_\Gamma G(P,Q)\frac{\partial}{\partial n}\phi(Q)\,d\Gamma(Q)\ , \quad (1)$$

where Γ is the boundary of the domain $\mathcal{D}$ and, the point source potential

$$G(P,Q) = -\frac{1}{2\pi}\log\|P-Q\| \quad (2)$$

can be employed for the Green's function. For the moment, it is convenient to restrict the point P to be inside the domain $\mathcal{D}$, and thus the coefficient 1 for the leading term in this equation. For $P \in \mathcal{D}$, the integrands are not singular, and the derivative of Eq. (1) respect to P can be computed by interchanging the order of differentiation and integration. The resulting boundary integral equation for the gradient of the potential can therefore be expressed as

$$(\nabla\phi\cdot D)(P) + \int_\Gamma \phi(Q)\nabla\frac{\partial}{\partial n}G(P,Q)\cdot D\,d\Gamma(Q) = \int_\Gamma \frac{\partial}{\partial n}\phi(Q)\nabla G(P,Q)\cdot D\,d\Gamma(Q), \quad (3)$$

where D is any specified direction vector.

The traditional method for obtaining the desired boundary formulation for Eq. (1) is to deform the boundary around $P_0 \in \Gamma$ and define the singular integral involving $\partial G/\partial n$ as a Cauchy Principal Value (CPV) (Brebbia et al.[4]). Somewhat more complicated variations of the CPV regularization procedure have also been employed to make sense of the *hypersingular* integral in Eq. (3) involving two derivatives of the Green's function (see the reviews by Krishnasamy et al.[1] and Lutz et al.[5]). An alternative method, successful for *both* equations (Gray,[6] Gray et al.[7,8]), is to define the singular integrals as a limit to the boundary, $P \to P_0$. With this definition, Eq. (1) and Eq. (3) *remain valid* as they stand for $P \in \Gamma$. Note that the interior boundary angle coefficient (interior solid angle in three dimensions) that is commonly present in writing Eq. (1) is automatically calculated within the limit process (Gray and Lutz[8]).

The corner analysis presented by Gray and Lutz[3] was for three-dimensions and the calculations employed a nonstandard interpolation algorithm. Moreover, the discussion was further complicated by the choice of limit process used to evaluate the singular integrals. In this paper, the issues involved in employing a hypersingular equation at a corner point will be considered for the simpler two-dimensional situation. In particular, an alternative, more straightforward, limit procedure for evaluating the singular integrals is described in the next section. Analytic formulas for the singular integrals are derived, and, unlike the previous approach, a single formula is valid for all elements comprising the corner. Together with the two-dimensional setting, this simplfies the discussion

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of the constraints imposed upon the boundary interpolations at the corner. In the subsequent section, these techniques are used to examine the 'L-shaped domain' potential problem discussed in Jaswon and Symm's classic text.[9] For this problem, traditional boundary integral methods do not yield accurate solutions in the vicinity of the corners.

HYPERSINGULAR INTEGRALS

Denote the boundary point P_0 by (x_0, y_0), and assume that the integration path is a straight line segment connecting P_0 to the point $P_1 = (x_1, y_1)$. The linear element is used solely to simplify the subsequent discussion, and it should be emphasized that higher order curved elements can also be handled (Gray[10]). In parametric form, the element to be integrated over is therefore

$$Q(t) = (x_0, y_0) + t\{(x_1, y_1) - (x_0, y_0)\} \ , \tag{4}$$

where $t \in [0,1]$. To compute the limit to the boundary, the interior point P in Eq. (3) is chosen as $P_\epsilon = P_0 + \epsilon L$, $L = (L_1, L_2)$, $||L|| = 1$. As $\epsilon \to 0$, $P_\epsilon \to P_0$ along the approach direction L, which is arbitrary as long as it is not tangent to the boundary. While the complete singular integration involving P_0 (in the case of two dimensions, there are two elements) is independent of the choice of L, the integral over a particular element *does depend* upon L.

At a corner point, there is cancellation of singularities between the two integrals in Eq. (3), and it is therefore important that these integrals be treated in a consistent fashion (Gray and Lutz[3]). However, for indicating the types of integrals that arise, it suffices to examine the hypersingular term. Differentiating Eq. (2), the hypersingular kernel is found to be

$$\frac{\partial}{\partial D}\frac{\partial}{\partial n}G(P,Q) = \frac{1}{2\pi}\left[\frac{n \cdot D}{r^2} - 2\frac{(n \cdot R)(D \cdot R)}{r^4}\right] \ , \tag{5}$$

where $r(t) = ||R(t)|| = ||Q(t) - P||$. The integral to be evaluated is therefore

$$\frac{a}{2\pi}\int_0^1 \phi(Q(t))\left[\frac{n \cdot D}{r^2} - 2\frac{(n \cdot R)(D \cdot R)}{r^4}\right] dt \ , \tag{6}$$

where the Jacobian factor is $a^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$. The numerator in the integrand, including the factor $\phi(Q(t))$, depends upon the particular approximation employed, but it can be assumed to be a polynomial in t. The distance function in the denominator is given by $r^2(t) = a^2t^2 - 2a(T \cdot L)t + \epsilon^2$, where T is the unit tangent vector $(x_1 - x_0, y_1 - y_0)/a$. Evaluation of the hypersingular integral therefore reduces to computing terms of the form

$$\lim_{\epsilon \to 0}\int_0^1 \frac{t^k}{(a^2t^2 + bt + \epsilon^2)^j}\, dt \ . \tag{7}$$

Appropriate formulas can be found in most integral tables, but, as indicated above, it is much easier to employ symbolic computation (Gray[10]) to carry out the integrations *and* the limit process. It is important to remark again that, unlike for a smooth surface, the limit $\epsilon \rightarrow 0$ of Eq. (6) *does not exist.* There is a singular term of the form $\log(\epsilon)$ which will only cancel out if the difference of the two integrals in Eq. (3) is considered. A consequence of the proof that the singular terms do vanish is that the interpolation of the potential at the corner must satisfy certain constraints. In brief, the two tangential derivatives of ϕ at the corner must be consistent with the normal derivatives. For most boundary element interpolations algorithms, the tangential derivatives are not directly considered, but are implicit in the polynomial approximation for ϕ. The details of this analysis are too lengthy to go into here and will be published elsewhere.

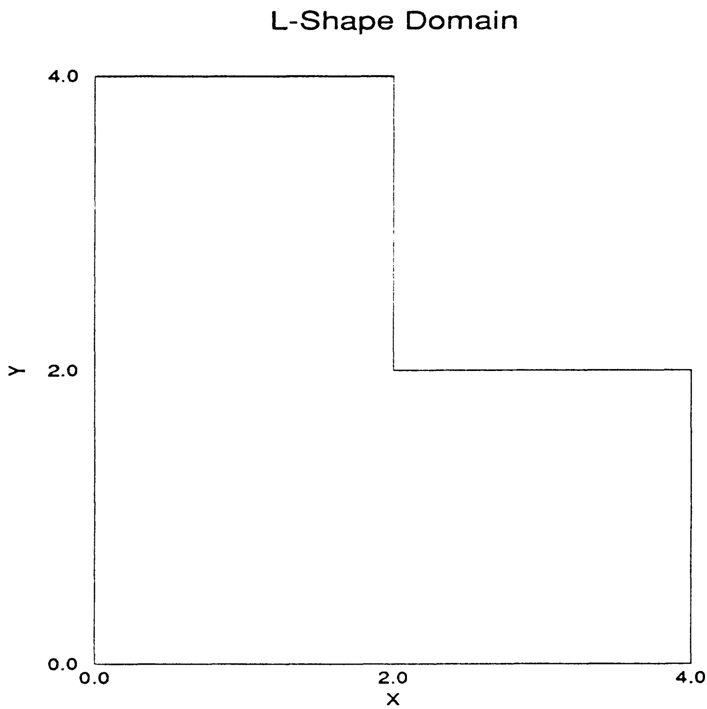

Figure 1: Jaswon and Symm's L-shape domain problem.

CALCULATIONS

The problem posed by Jaswon and Symm[9] is a simple Dirichlet boundary value for the L-shape domain shown in Figure 1. The applied boundary condition is

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$\phi(x, y) = x^2 - y^2$, and thus the normal derivative to be calculated everywhere on the boundary is $(2x, -2y)\cdot\vec{n}$, where $\vec{n}$ is the boundary normal. The boundary arcs are parallel to the axes, and thus the exact solution is constant on each boundary segment. At the boundary corners, there are two normal derivative values to be determined, and the solution vector is discontinuous through the corner.

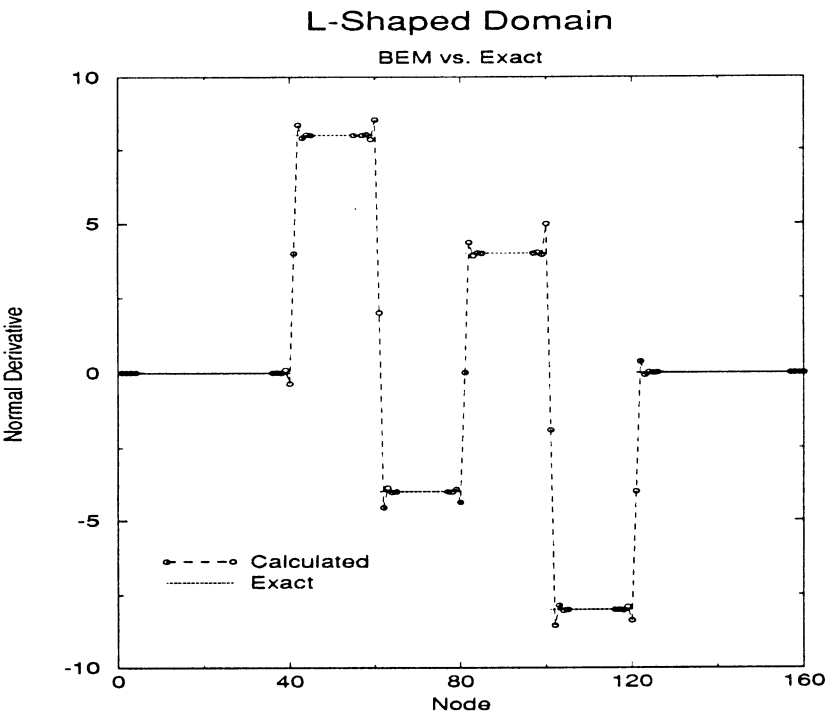

Figure 2: Exact vs. calculated normal derivative for the L-shape domain.

As discussed extensively by Jaswon and Symm,[9] a straightforward boundary element formulation of this problem, one without any special consideration of the corners, will yield an inaccurate solution. Figure 2 displays the results of such a calculation employing 160 equally spaced nodes and an Overhauser spline approximation (Camp and Gipson[11]). Except for the vicinity of the corners (nodes $1, 41, 61, 81, 101, 121$), the boundary element solution is virtually the same as the exact solution (the solid horizontal lines), and thus only the calculated values near the corners are plotted. Note that the boundary element solution is similar to a Fourier series approximation to a discontinuous function. At the corners, the calculated value is nearly the average of the two correct values on either side of the corner, while the solution oscillates (Gibbs phenomenon) in the neighborhood of these points.

As described previously (Gray and Lutz[3]), the extra unknown at the corner can be handled by writing an additional equation at these points. The boundary integral equation Eq. (1) can be combined with Eq. (3) or, as demonstrated herein, two hypersingular equations can be employed. If N_1 and N_2 are the two normals at the corner, the equations

$$\begin{aligned} \nabla\phi \cdot N_1 &= \int_\Gamma \left[\frac{\partial}{\partial n}\phi(Q)\nabla G(P,Q) - \phi(Q)\nabla\frac{\partial}{\partial n}G(P,Q)\right] \cdot N_1\, d\Gamma \\ \nabla\phi \cdot N_2 &= \int_\Gamma \left[\frac{\partial}{\partial n}\phi(Q)\nabla G(P,Q) - \phi(Q)\nabla\frac{\partial}{\partial n}G(P,Q)\right] \cdot N_2\, d\Gamma \qquad (8) \end{aligned}$$

are independent and will effectively determine the corner values.

As indicated above, the limit direction L used for evaluating the singular integrals can be chosen arbitrary. In fact, as a check, two calculations were performed, the only difference being the value of L, and there was essentially no change in the solution. In the results reported below, L is along the bisector of the interior angle at the corner.

The calculation reported below employs the Hermite approximation described in Gray and San Soucie[12] (this type of approximation for boundary integral analysis was first investigated by Watson[13]). This algorithm incorporates the tangential derivative of the potential along the boundary, and these quantities are determined by Eq. (3) with $D = T$. As discussed above, the tangential derivatives at a corner are determined from the normal derivatives, and thus, the tangential hypersingular equations are not needed at these points. This is fortunate, as the tangential equations and Eq. (8) would be dependent. There are two advantages to the Hermite approximation for this problem. In the first place, a linear geometry can be employed in conjunction with a high order (cubic) approximation for ϕ. As indicated above, the linear element simplifies the task of evaluating the singular integrals. More importantly, as mentioned above, the cancellation of singularities in the hypersingular equation at a corner imposes constraints on the interpolation of the potential (Gray and Lutz[3]). As these constraints involve the tangential derivative of the potential, they are easily incorporated into the calculation using the Hermite approach.

The primary concern herein is the treatment of the corner, so one other non-standard aspect of the calculation will only be mentioned in passing: the boundary integral equation Eq. (1) has not been used at all. Instead, the hypersingular equation is employed everywhere on the boundary. The rationale for this approach is discussed elsewhere (Gray et al.[14]), where it is shown that a hypersingular formulation is particularly effective for Dirichlet boundary value problems.

Figure 3 plots the error (scaled by the factor 10^{12}) in the computed normal derivative, for the same discretization employed in the standard boundary ele-

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ment calculation (Figure 2). However, to account for the two solution values, two nodes are employed at the corners, and thus there are now 166 nodes in the model (the corner double nodes are numbered $161-166$). Note that the actual geometry and the element approximation are linear. Similarly, the exact potential and normal derivative functions on the boundary are respectively quadratic and piecewise constant. The Hermite interpolation employs a cubic polynomial for ϕ and a linear approximation for the flux, and thus neither the geometry nor the function interpolations introduce any error in the calculation. The only source of error is therefore the numerical quadrature (a 4 point Gauss rule) used for the nonsingular integrals, and it is therefore not surprising that the computed solution is essentially exact (maximum error $= 0.585 \times 10^{-12}$) everywhere on the boundary. One cannot, in general, expect results this accurate. However, as desired, this calculation clearly demonstrates that the hypersingular equation at the corners is being evaluated correctly.

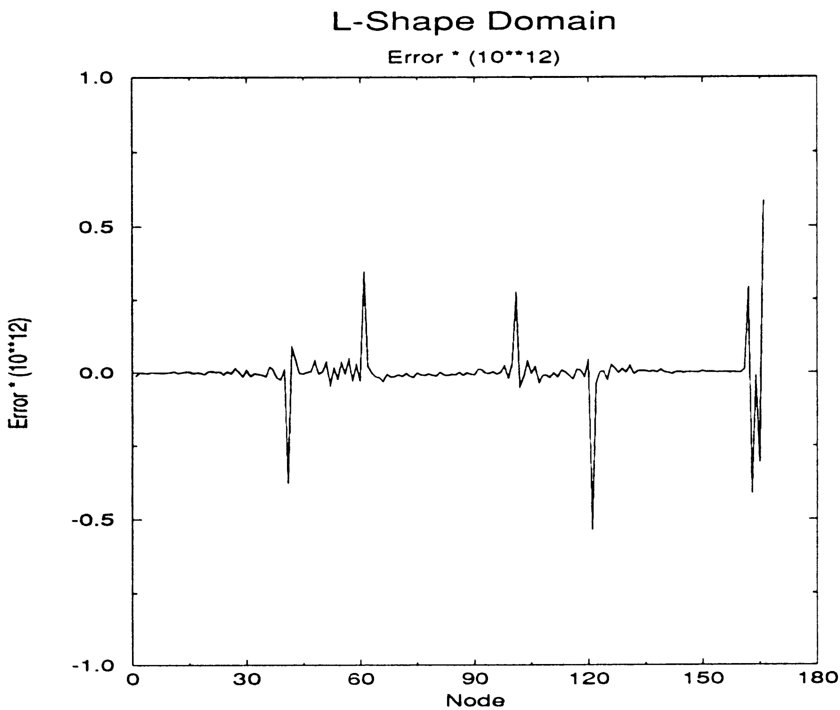

Figure 3: Error in the computed normal derivative for the L-shape domain problem using the hypersingular formulation.

It is conceivable that the accuracy at the corners is fortuitous, a consequence of having only 90^o corners, in which case the corner constraints equations are quite simple. The same boundary value problem was therefore examined for the triangular domain with vertices at $(0,0)$, $(4,0)$, and $(0,4)$. The calculation was again highly accurate, the maximum error in the normal derivative in this case being 0.441×10^{-8}. Figure 4 plots the error (scaled by a factor of 10^8) as a function of node number (the numbering scheme starts at $(0,0)$ and proceeds counterclockwise around the triangle, the three double corner nodes once again being the last three nodes).

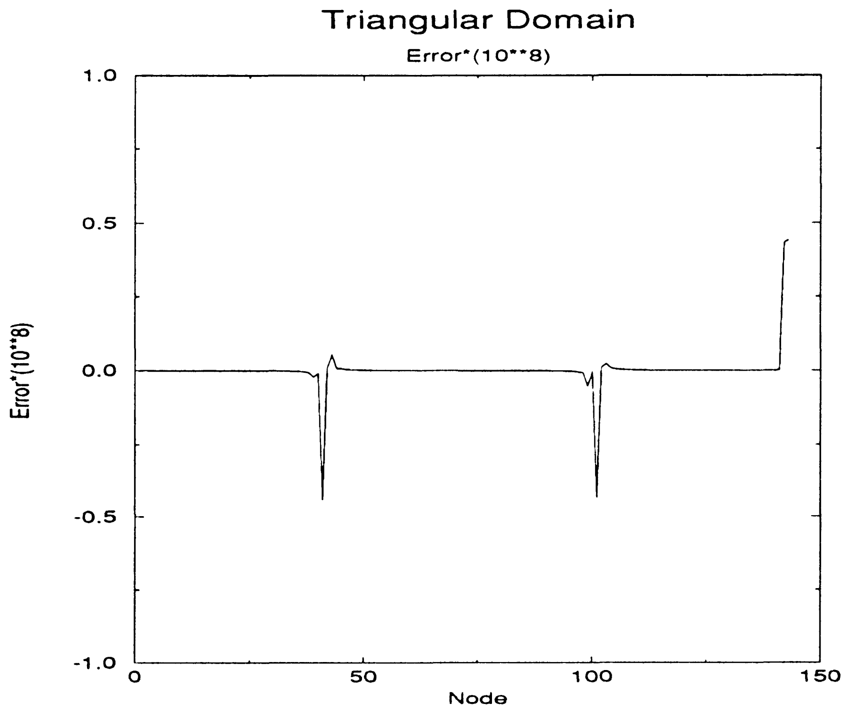

Figure 4: Error in the computed normal derivative for the triangular domain problem using the hypersingular formulation.

CONCLUSIONS

As shown previously in three-dimensions, hypersingular equations can be employed at a boundary corner. For the example problem investigated herein, the boundary normal derivative is discontinuous at the corners due solely to the change in normal direction. That is, the interior gradient *is continuous* to the boundary. Under these circumstances, the hypersingular integral can be defined using an arbitrary limit direction, and this can be used to advantage for boundary corner points. The analytic integration formulas for the singular integrals are necessarily more complicated than for approaching normally to the boundary, but can be easily derived using symbolic computation.

In contrast to a smooth surface, the hypersingular and flux integrals in Eq. (3) do not exist at a boundary corner. However, provided the interpolation at the corner is consistent, the difference of the two integrals does have a finite limit at the boundary, and the hypersingular equation is meaningful.

ACKNOWLEDGMENTS

This work was supported by the Applied Mathematical Sciences Subprogram, Office of Basic Energy Sciences, U.S. Department of Energy, under contract DE-AC05-84OR21400 with Martin Marietta Energy Systems Inc.

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