A boundary element method approach to the vibration problem of stiffened plates

K. Toumi, L. Jezequel

Ecole Centrale de Lyon, Laboratoire Mécanique des Solides, URA CNRS 855, BP 163.69131 Ecully, France

1: INTRODUCTION

In many cases, the Boundary Element Method (BEM) appears as the best alternative to solve physical problems. The number of studies on the BEM have increased sharply during the past few years, in order to adapt it to industrial problems. Many real structures, such as ships, big transformers, etc., are made of assembled plates and/or stiffened panels. Various methods can be used to predict the dynamic behaviour of the global structures. With methods like the Finite Element Method, a lot of time is generally spent on the preprocessing phase to find a correct model for the structure. This time is generally ten to twenty times longer than the computation time. As the BEM only needs the description of the structure's boundary, the preprocessing work is considerably diminished. Moreover, as we use a mixed formulation, equilibrium is globally and locally satisfied. This property is very important in the case of stiffened plates, especially along the connecting lines.

After a brief description of the BEM's application to a simple plate for bending vibration and inplane motion, we will present the procedures we can use to connect the movements of the assembled plates. For this purpose, we will introduce the line variable notion and show how it seems to be better suited to the BEM formulation than the classical nodal variable. Each line is then described by discretized integral equations, boundary conditions and finally, if necessary, coupling equations. In the case of singular points (the intersection of boundary lines), which introduce additional unknowns, we will present suitable equations so as to have as many equations as unknowns. The special and essential case of stiffened plates will be discussed with great attention.
The integral formulation of the dynamic behaviour of a plate in bending and inplane motion consists in the equations:

\[
\int_{\Gamma} \left( V_\alpha^{*} (w - w^p) - M^{*} N + N^{*} M - w^{*} V \right) d\Gamma + \sum_{i=1}^{n} F_i^{*} (w_i - w^p) - w^{*} F_i = 0 \\
\rho h \int_{\Omega} w^{*} \ddot{w} d\Omega
\]

\[
\int_{\Gamma} \left( p_{ij}^{*} (u_j - u_j^p) - u_{ij}^{*} p_j \right) d\Gamma - \rho h \int_{\Omega} u_{ij}^{*} \ddot{u}_j d\Omega = 0; \quad i, j = 1, 2
\]

where \( \Omega \) and \( \Gamma \) represent the internal domain and the boundary respectively. \( P \) is the source point to which the integrals are applied. \( w^{*}, N^{*}, M^{*}, V^{*} \) are the integral kernels defined on the boundary. These kernels result from applying Green's identities on the static bending operators. They are well explained in [Bézine\(^4,5\), Hartmann\(^9\), Stern\(^15\), Toumi\(^17,18\)]. \( w, N, M, V, F \) represent, respectively, the deflection, normal rotation, normal moment, equivalent shear and the torsional moment discontinuity in the corner. In the following we will replace \( F \) by its development along \( N \). \( u_{ij}^{*}, p_{ij}^{*} \) are the integral kernels associated to the plane stress equations; they are developed in [Brebbia\(^6\), Banergee\(^3\), Toumi\(^17,18\)]. \( u_1, u_2 \) (resp \( p_1, p_2 \)) are the inplane displacements (resp. the traction vector components) along the \( x:1 \) and \( y:2 \) axis. Equations (1) are written for source boundary points \( P \). Similar integral equations are used to calculate the displacements or efforts on an internal point. These equations can be found in the above-mentioned references.

To compute the domain integrals in the three equations 1-a and 1-b, we consider a set of distributed internal points. These points are the centers of gravity of the domain's sub-dividing cells. Each point has a weight equal to the area of the corresponding cell. The development of the domain integrals along these points with the corresponding weight coefficients gives us an estimation of the inertial distribution on the plate. To have more information on this approach, the reader can refer to (Katsikadelis\(^13\) and Toumi\(^17,18\)).

These integral equations are then associated with boundary conditions. For each regular point, there are two boundary conditions for bending and two others for plane stress. Let us consider the example of a plate having an edge that cannot move in
its normal direction but is free in the tangential one. The equations which express the boundary conditions for inplane motion are: \( \cos \alpha \cdot u_1 + \sin \alpha \cdot u_2 = 0 \) and \( -\sin \alpha \cdot p_1 + \cos \alpha \cdot p_2 = 0 \), where \( \alpha \) is the angle between the normal to the edge and the \( x \) axis. Using variables \( u_n, u_t, p_n, p_t \), the last two equations can be simply written as \( u_n = 0 \) and \( p_t = 0 \). But, in the matricial formulation stage, adding new equations to take the boundary conditions into account is not an efficient solution. It would be better to reduce the number of unknowns resulting from these conditions. To do so, the normal and tangential variables are the most suitable. In this paper we will use the \( u_n \) and \( u_t \) variables for the displacements and \( p_n \) and \( p_t \) for the traction vector. The two equations 1-b are replaced by similar ones where \( i,j \) become \( k,l \) and where \( k,l \) take their values from \{n,t\} as \( i,j \) took them from \{1,2\}. The integral kernels \( u_{ij}^* \) are replaced by other kernels \( u_{kl}^* \) given by:

\[
\begin{align*}
u_{kl}^* &= l_i \cdot u_{jl}^*, \\
p_{kl}^* &= l_i \cdot p_{jl}^*, \\
k, l &= n, t; \quad i = 1, 2, \quad j = 1 (\text{resp} 2) \quad \text{if} \quad k = n (\text{resp} t) \quad (2)
\end{align*}
\]

Usually, when we have written the integral equations -coupled with the boundary conditions- we can discretize the boundary. This leads to an element and node description. This approach greatly resembles the FEM; an element library can also be created, containing, for example, the linear two-node element, the quadratic element, etc, for plate problems. In the next chapter, we will introduce the notion of line variable; this will enable us to depart from the definition and organization of the elements.

By using suitable interpolations for all the boundary variables, integral equations 1-a and 1-b are converted to linear algebraic equations. These concern the boundary unknowns and internal displacements. The technical procedures used to compute eigenvalues and modes are well described in (Katsikadelis\textsuperscript{3} and Toumi\textsuperscript{17,18}).

**III. LINE VARIABLE NOTION:**

It is clear that the first and principal difference between the FEM and the BEM lies in the modelling. The integral method is based on a model which is close to the geometrical construction. Using it is thus obviously easier for the computation code user. For assembled plate structures, the geometry is entirely defined from the boundary lines of each plate. The basic idea is to define each plate thanks to these lines alone. Each line is then defined on the one hand by the geometrical points and on the other hand by computational points which are independent of the first ones. Line
geometry can be defined by a minimum number of geometrical points (two in the case of a straight line). The number of computational points of each line will depend on the precision required. The introduction of these computational points can be automatic; their number and position could also be determined by an error estimation procedure. This part of the study is not treated here. The numbering of the computational points is local: it is related to each line. From a global point of view, we take into account only the line numbers and their corresponding geometrical points.

$P_i; i=1,n$: geometrical points; $Q_j; j=1,m$: computational points

*fig 1- a- common case: straight line; -b- general case: curve line*

In a computational process, this line description allows easy use of variable interpolations with high degree polynomials, for instance the Lagrange ones. In an extreme case, we could use $(m-1)$ degrees of interpolation for a line with $m$ computational points. In practice an interpolation with a limited degree, $p$, is sufficient. Such an interpolation with high degree polynomials enables us to obtain a good numerical estimation of the boundary integrals; this increases the precision of the final results.

During code conception, using the line variable also makes it easier to manage the data arrangement. Indeed, figure 2 represents two structures made up of assembled plates. In the first case, we write the coupling equations between $S_1$ and $S_2$, $S_2$ and $S_3$ and finally between $S_3$ and $S_1$. In the second case, coupling exists only between $S_1$ and $S_2$ and between $S_2$ and $S_3$. In both cases, however, the corner point, $P$, belongs to all three plates. In the second case, we cannot write the coupling between $S_1$ and $S_3$, although $P$ belongs to both plates. In fact, the coupling is realized whenever an entire line
is common to two plates. In the next paragraph, we will describe the coupling equations.

Fig 2: Plate coupling around a corner point

IV: PLATE ASSEMBLING:

We are concerned with assembling out-of-plane plates and not only inplane ones. The question is to couple the bending and inplane motions of each plate with the other plates' motions. Before writing the coupling equations, we shall specify that a plate assembled in a structure is treated as a simple one; the difference is that when an edge is common to another plate, the boundary conditions are replaced by the coupling equations. On each regular point, we obtain as many equations as unknowns. We will see in the next chapter that the case of singular points is more complicated to solve.

Let us now consider the case of several plates with a common line $L_c$ (figure 3); all the boundaries of the plates except $L_c$ can have any boundary or coupling conditions. Let us consider a plate $S^i$; for each point $P$ in line $L_c$, the degrees of freedom that describe the bending and inplane motion are $u^i_*, u^i_t, w^i$ for the displacements, $N^i$ for the normal rotation, $M^i$ for the normal moment and finally $p^i_*, p^i_t, V^i$ for the efforts. $R^i_c$ is the rotation matrix between the two local coordinate systems $(n,t,z)$ -which is related to line $L_c$ -and $(x,y,z)$ -which is related to plate $i$. Let us call $Q^i$ the transformation matrix from the local system $(x,y,z)$ to the global system $(X,Y,Z)$.

$$U^i_p = [u^i_*, u^i_t, w^i]; \quad F^i_p = [p^i_*, p^i_t, V^i]$$

(3)
On each point $P$ of line $L_c$, the coupling equations express:

- The displacement continuity:

$$ Q^i.R_c^i.U^{ip} = Q^i.R_c^{iP}; \quad i = 2, ns $$

(4)

where $ns$ is the total number of plates to which $L_c$ belongs

- The continuity of the rotation around $L_c$:

$$ \varepsilon_i.N^{ip} = N^{iP}; \quad i = 2, ns $$

(5)

$\varepsilon_i=1$ if plates 1 et $i$ are identically oriented on $L_c$ and $\varepsilon_i=-1$ otherwise.

- The effort equilibrium:

$$ \sum_{i=1, ns} Q^i.R_c^i.F^{ip} = 0 $$

(6)

- The equilibrium of the moments around $L_c$:

$$ \sum_{i=1, ns} \varepsilon_i.M^i = 0; $$

(7)

More than the displacement continuity, these equations induce effort equilibrium around the assembling line. This property is of major importance for a stiffened plate study. Indeed, many welded plate structures fail in the vicinity of the weld during their life. A crack can easily appear wherever there is acute stress concentration. Thus it is essential to evaluate the stress variation along the
soldered line as precisely as the displacement. The other methods, like FEM, evaluate the stresses only during the postprocessing stage. It is well known that with FEM the stress field continuity between elements is not satisfied. Moreover, this field is not well-balanced at each point of the line. For all these reasons, the estimation of stress is never as precise as that of the displacements. To get round these inaccuracies, two solutions are usually used. The first one consists in introducing fine meshing near the assembling lines. This approach can become very expensive with the number of stiffeners. The second one consists in finding, during the postprocessing phase, a locally well-balanced field which no longer satisfies the stress-strain relationships. Although this method can be less expensive than the first one, stress accuracy remains inferior to displacement accuracy. Lastly, let us remember that the integral formulation principle implies the global equilibrium of each plate and consequently, as local equilibrium is satisfied along each connection line, so is the equilibrium of the whole structure.

IV SINGULAR POINTS:

For a simple plate, a singular point refers to any point for which both tangent lines and/or both boundary conditions and/or both loads to its right and to its left are different. Under some boundary conditions, this singularity induces a lack of equations related to the number of unknowns. For this reason, some researchers have proposed additional equations in bending or in plane stress. Here we want to generalize the singular point definition to the case of a plate assembly and to propose a strategy for generating additional equations with a view to obtaining a final system made up of unrelated equations. Figures 4-a and 4-b present two simple examples of assembled plates. In the first case, no boundary conditions but coupling equations are to be written for the singular point. The equations which express the displacement continuity are:

\[
\begin{align*}
Q^1 \cdot R^1 \cdot U^{1p} &= Q^2 \cdot R^2 \cdot U^{2p} \\
Q^2 \cdot R^2 \cdot U^{2p} &= Q^3 \cdot R^3 \cdot U^{3p} \\
Q^3 \cdot R^3 \cdot U^{3p} &= Q^1 \cdot R^1 \cdot U^{1p}
\end{align*}
\]  

(8)

This system is composed of related equations; its introduction in the global matrix will make it non reversible. Thus we should draw one equation from the system and replace it by another one, unrelated to any of the other equations written for the structure.
Plate assembly, in the second example (4-b), requires coupling equations and boundary conditions; some of them can be written:

\[
\begin{align*}
U^{2p} &= 0 \\
U^{3p} &= 0 \\
Q^1 R^1 U^{1p} &= Q^2 R^2 U^{2p} \\
Q^2 R^2 U^{2p} &= Q^3 R^3 U^{3p} \\
Q^4 R^4 U^{1p} &= Q^5 R^5 U^{1p}
\end{align*}
\]  

(9)

This system is not free but this time, two of the five equations should be replaced by two of the three equations left. The third example is even more complicated and it becomes difficult to determine the number and kind of equations that make the system not free. To solve this problem, two phases are required: the first one consists in duplicating the singular node (as many times as there are lines which contain it) and the second one consists in drawing a graph which describes the different potential equations.

**Duplication of a singular point:**

As we describe a plate through its lines, we can assess that its internal computational points are all regular. Only the two extreme points can be singular. To simplify the manipulation of these points, we will suppose they are all singular. This is not really constraining. Let us consider any singular point which belongs to \(N_l\) lines and \(N_s\) plates. Until now, each plate had its own computational point in \(P\) (see figure 5-a). We then had \(N_s\) points. We now replace them by \(N_l\) other computational points: one point per line (see figure 5-b). This duplication calls for some remarks:
- The geometrical position of the computational points remains the same as that of the initial singular point.

![Diagram of computational points](image)

**Fig 5: computational points-a: relative to the plate-b: relative to the lines**

- Even if a line, containing the singular point, belongs to several plates, only one computational point is created. This means that the coupling conditions between plates have been already satisfied. On the contrary, when a singular point belongs to a plate, it automatically belongs to two lines of this plate; thus, two computational points are created.

- For each plate, the duplication of the singular points on each corner does not imply the duplication of the integral equations because the source point is the same.

- Both the boundary conditions and the coupling equations written on the line to the left of the point are not related to the ones written on the right.

- To close the system, we must write "closing equations". For each plate, these equations express the coupling between the variables from the left line and the right line. They also ensure displacement continuity at the singular point. If we consider again the example illustrated in figure 4-b, the system (9) becomes, when the closing equations have been added:

\[
\begin{align*}
U^{1p} &= 0 \\
U^{4p} &= 0
\end{align*}
\qquad \text{and} \qquad
\begin{align*}
R_1^1 U^{1p} &= R_2^1 U^{2p} \\
R_2^2 U^{2p} &= R_3^2 U^{2p} \\
R_3^3 U^{3p} &= R_4^3 U^{4p}
\end{align*}
\]

Systems (9) and (10) are composed of linearly dependent equations.
In system (10), the coupling equations no longer appear and boundary conditions and closing equations have been separated. Whenever the system is not free, we should replace a closing equation by another, unrelated to the other ones; the boundary conditions are always expressed whatever the nature of the system. When the number of lines containing the singular point increases, this operation becomes very hard to realize. Its automation requires the use of the graph theory.

**Graph of a Singular Point:**

Let us consider once again the singular point belonging to \( NL \) lines \( L_1, L_2, \ldots, L_{NL} \) and \( NS \) plates \( S_1, S_2, \ldots, S_{NS} \).

Each line \( L_i \) represents a "node" \( i \) of the graph. We add to these nodes node 0 whose usefulness will be shown later.

We will call "link" the connection between two nodes of the graph. Between two different nodes \( i \) and \( j \) we establish a link only if the two corresponding lines \( L_i \) and \( L_j \) belong to the same plate \( S_k \).

Graph \( G \) is entirely defined by its nodes and its links.

A "way" is a succession of links continuously connecting two nodes.

A "loop" is a particular way which connects a node to itself.

A way is considered as "closed" if it contains at least one loop, otherwise, it is called "open".

A graph is considered as "closed" if it contains at least one closed way, otherwise it is called "open".

Fig 6: graphs of two singular points (structures Fig 4-b and 4-c).
Figure 6 shows the graphs of the singular points of the structures represented in figures 4-b et 4-c. Graph 6-a contains a loop, thus at least one closed way. If we break only one link, the graph becomes open. In figure 6-b, we have to break two links to open the graph.

To each node \( L_i \) of the graph we associate a variable \( x_i \), to each link \( (L_i, L_j) \) a linear equation \( E_{ij} \) relating the two variables \( x_i \) and \( x_j \). The obtained set of equations \( E_{ij} \) produce a linear algebra system \( E \). We can demonstrate the following property:

**Graph G is open if and only if E is free**

In the case of bending, node 0 represents clamping, the \( x_i \) variable the deflection \( w_i \) of the computational point of the line containing the singular point. If one of the node is equal to zero (i.e. if the other line is clamped), equation \( E_{ij} \) represents a boundary condition (bold line on the graphs in figure 6). Otherwise, \( E_{ij} \) represents a closing equation (fine line). In this case, the closing equation is: \( w_i = w_j \) between lines \( L_i \) and \( L_j \) belonging to the same plate. We can easily verify that the two systems of equations corresponding to the graphs in figure 6 are not free. To set them free we will break the links between some nodes in order to open the graphs. To describe all the boundary conditions we will never break a link for which a node is equal to zero. Each broken link will be replaced by an equation relating to another variable \( w \). Thus we keep the same number of equations as unknowns and we make sure the system is free.

In the case of inplane motion, there are two displacement variables, so we cannot apply the graphs in figure 6 directly. We have to create two graphs: one for displacement \( u_1 \) and the other for \( u_2 \). Then we have to create the links between the different nodes of these graphs. The inter-graph links reflect the boundary conditions related to displacements \( u_n \) and \( u_t \). In making these links, we obtain one graph to which we can apply the same procedure as for bending. The additional equations which replace the broken links appear in the appendix.

V- EXAMPLE:

Let us consider a symmetrical structure composed of four stiffened panels. For symmetry reasons we will model only a quarter of the structure. So we will consider two orthogonal half-panels. Each panel is stiffened in two orthogonal directions. The stiffeners are plates of different shapes. We can find this kind of stiffener in ships, for example.
The Boundary Integral model is thus composed of 32 plates. Each plate is then bounded by 4 or 5 lines and each line is described by 2 geometrical points. The number of computational points depends on the length of each line: it varies between 3 and 20. Points like numbers 4, 5, 6 belong to many plates and lines at the same time but their graphs are open; so we do not use additional equations to replace some closing equations. However, points like numbers 1, 2 or 3 have closed graphs and so we have to apply the above-mentioned procedure to detect the closing equations which will be replaced by additional ones. Figure 7 represents the first two modes of this structure.

VI-CONCLUSION:

In this paper we have introduced the line variable notion. It brings the boundary integral model closer to the geometrical one. The use of high degree polynomial interpolation also becomes easier. As we never use the usual elements, the program organisation becomes more simple. We have written the coupling equations for each point of the common lines. These equations express displacement continuity and effort equilibrium. The second property is very important in the case of stiffened plates. In fact both the local and global plate equilibrium are satisfied. For each singular
point, we create one computational point per line. To satisfy the
displacement continuity on the corner we write closing equations.
These equations can, in some cases, make it impossible to invert the
final matrix. To get round this problem we build a graph for each
singular point. This graph indicates the number and the nature of
the closing equations we should replace by additional ones (see
appendix).

VII- REFERENCES:

1  ABDEL-AKHER A. & HARTLEY G. A. 'Evaluation of boundary
integrals for plate bending'. Int. Journ. num. meth eng, vol 28,
75-93 (1989).
2  ABDEL-AKHER A. & HARTLEY G. A. 'Boundary integral and
interpolation procedures for plate bending' Int. Journ. num.
3  BANERGEE P. K. AND BUTTERFIELD R. Boundary Element
4  BEZINE G. 'A New Integral Equation Formulation for Plate
Bending Problems'. 1978. Recent Advances in Boundary
Element Methods 327-342. Ed. C. A. BREBBIA
5  BEZINE G. 'A mixed boundary integral finite element approach
to plate vibration problems.' 1980 Mechanics Research
Communications 7, 141-150
6  BREBBIA C.A. AND DOMINGUEZ J. Boundary Elements: An
Introductory Course. Mc Graw-Hill.
7  CHAUDONNERET M. 'On the Discontinuity of the Stress Vector
in the Boundary Integral Equation Method for Elastic Analysis'.
1978. Recent Advances in Boundary Element Methods
8  BESCKOS D. Boundary element methodss in dynamic
analysis.App Mech Rev vol , no 1, Jan 1897.
9  HARTMANN F. AND ZOTEMANTEL R. 'The Direct Boundary
10  BALAS J., SLADEK J. & SLADEK V., Stress Analysis by Boundary
Element Method, Elsevier Amsterdam, Oxford, New York,
Tokyo 1989.
11  KAMIYA N. AND SAWAKI Y. 'An Integral Equation Approach to
Finite Deflection of Elastic Plates.' Int Jour. of Non-Linear
12  KITAHARA M. 1985 Boundary Integral Methods in Eigenvalue
Problems of Elastodynamics and Thin Plates. Amsterdam:
Elsevier.
13  KATSIKADELIS J. T. 'Boundary Element solution to the
APPENDIX

The additional equations that can replace the closing ones concern the dual displacement variables. We propose here 2 equations for bending:

$$\cos(\alpha^+ - \alpha^-)(M_n^+ - M_n^-) - \sin(\alpha^+ - \alpha^-)(M_n^+ + M_n^-) = 0 \quad (11)$$

$$\left(1 - \cos(\alpha^+ - \alpha^-)\right)\left(V_n^+ + V_n^- + D(1 - \nu)\left(\frac{\partial^2 N^n}{\partial t^2} + \frac{\partial^2 N^n}{\partial t^3}\right)\right) +$$

$$\sin(\alpha^+ - \alpha^-)\left(M_n^+ - M_n^- - D(1 - \nu)\left(\frac{\partial^3 w^+}{\partial t^3} + \frac{\partial^3 w^-}{\partial t^3}\right)\right) = 0 \quad (12)$$

and for plane stress:

$$\cos(\alpha^+ - \alpha^-)(p_n^+ - p_n^-) - \sin(\alpha^+ - \alpha^-)(p_i^+ + p_i^-) = 0 \quad (13)$$

$$\frac{1 + \nu}{E}(p_n^+ - p_n^-) + \frac{\partial u_i^+}{\partial t} - \frac{\partial u_i^-}{\partial t} = 0 \quad (14)$$

$x^-$ (resp $x^+$) refers to the value of $x$ on the left (resp right) line.

It is possible to use other kinds of equations but these ones suffice to solve all the configurations.