Boundary element eigenvalue analysis by standard routine
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ABSTRACT

This paper presents a method for determining eigenvalues of the Helmholtz equation by using "standard subroutine". The formulation is based on the boundary integral equation with a help of Multiple Reciprocity Method (MRM). The resulting polynomial coefficient matrix is shown to be transformed into the standard-type eigenvalue problem. Following the proposed scheme, we can obtain eigenvalues by "boundary discretization" using boundary elements and "standard eigenvalue determination routine" offered as blackbox.

INTRODUCTION

The ordinary boundary element formulation for analyzing eigenvalues of the Helmholtz equation using its fundamental solution yields coefficient matrices including unknown eigenvalue implicitly (called λ-matrices), for which it is generally hard and combersome to obtain the eigenvalues from the zero of their determinant [11]. The most popular method for the formulation employs direct determinant search using small increments of the
eigenvalue and iteration if required. Its disadvantages are lower computational efficiency and requirement for information on rough estimation.

From the viewpoint of efficient numerical calculation, user's load should be decreased. Boundary element method is superior in the angle of easy discretization; discretization is of main burden in preprocess of principal numerical tools requiring appropriate discretization either of domain by finite elements or boundary by boundary elements. Furthermore, we mention the use of existing standard routines for usual arithmetics such as linear simultaneous equation, numerical integration among others; eigenvalue analysis also falls into the such category.

The objective of the present paper is to derive a formulation which can be compatible on the ordinary subroutines for the eigenvalue determination in the framework of boundary discretization alone. The Dual Reciprocity Method (DRM) for the eigenvalue analysis [2-4] is the one satisfies the above requirements, whereas it uses additional internal points and special influence-type function. In what follows, we will show that the MRM formulation [5-7] can be transformed into the desired form using the property of \( l \)-matrices.

MULTIPLE RECIPROCITY METHOD FOR THE HELMHOLTZ EQUATION

The MRM was first proposed by Nowak and Brebbia [6] in order to convert domain integral to the equivalent boundary integral and applied further to the boundary-value problem with the Helmholtz equation [5]. The latter formulation was the base for the eigenvalue analysis by the present authors [6, 7]. First, we will show outline of the formulation.

Consider the following scaler-valued Helmholtz equation in terms of the potential \( u \) with the wavenumber \( k \):

\[
\nabla^2 u + k^2 u = 0
\]
defined in the domain \( \Omega \) in the two- and three-dimensional space bounded by the boundary \( \Gamma \).

According to the MRM [5-7] for the Helmholtz equation (1), the integral equation becomes as

\[
c u + \sum_{j=0}^{\infty} (\mathbf{H}_j) \mathbf{u} = \mathbf{G}(\mathbf{q})
\]

where the domain integral has dropped for its convergency [6]. In Eq. (2), \( q \) denotes the derivative \( \partial u / \partial n \) with respect to the outward unit normal \( n \) on the boundary and \( c \) is a constant determined by the geometrical condition of the boundary. \( u^* \) and \( q^* \) are the higher-order fundamental solutions for the Laplace equation, obtained by following equations:

\[
\begin{align*}
\nabla^2 u_0 + \delta &= 0 \\
q_0^* &= \partial u_0^* / \partial n
\end{align*}
\]

\[
\begin{align*}
\nabla^2 u_{j+1}^* &= u_j^* \\
q_j^* &= \partial u_j^* / \partial n \quad (j = 0, 1, 2, \ldots)
\end{align*}
\]

These are expressed, in terms of \( r \) the distance between the source and field points, as

\[
\begin{align*}
u_0^* &= -(1/2\pi)(1/4\pi(j!)^2)lnr - \frac{j}{(j!)}(1/e) \\
u_j^* &= 1/(4\pi r) \cdot 1/(2j)! \cdot r^{2j}
\end{align*}
\]

The discretized equation of Eq. (2) by boundary elements becomes as

\[
[H(k)]\{u\} = [G(k)]\{q\}
\]

where \( \{u\} \) and \( \{q\} \) are vectors constructed by \( u \) and \( q \) on the boundary nodes (N-th order vector for N nodes), \([H]\) and \([G]\) are defined as

\[
[H(k)] = \sum_{j=0}^{\infty} (-k^2)^j H_j, \quad [G(k)] = \sum_{j=0}^{\infty} (-k^2)^j G_j
\]
where \([H_j]\) and \([G_j]\) are computable on the boundary elements by integrating the fundamental solution multiplied by appropriate interpolation function.

Substituting the specified homogeneous boundary condition into Eq. (7), we obtain, with respect to the vector \(\{x\}\) composed of nonzero values of \(u\) and \(q\) on the nodes, the eigenvalue problem,

\[
[A(k)]\{x\} = [B(k)]\{0\}
\]

The proper equation is

\[
\text{det}[A(k)] = 0
\]

where, as can be seen from Eq. (8),

\[
[A(i)] = [A_0] + i[A_1] + \cdots + i^n[A_n]
\]

where \(i = k^2\). The matrices \([A_i]\)'s coincide with either \([H_i]\) or \([G_i]\) depending on the boundary condition. In [6, 7], the eigenvalues have been determined with the aids of the present formulation by the Newton iteration and LU decomposition following the idea by Yang [9]. Since \([A_i]\)'s are separated perfectly from the unknown \(i\), they are unnecessary to be recomputed in the iteration process, which reduced computational task significantly.

One shortcoming of the previous method is requirement for the specification of initial rough estimation as the start of iteration.

CONVERSION TO STANDARD EIGENVALUE PROBLEM

So-called standard eigenvalue problems and/or generalized eigenvalue problems are offered in the following forms:

\[
[A]\{x\} = i\{x\}
\]
where, coefficient matrices $[A]$ and $[B]$ in Eqs. (12) and (13) are not same to those in Eq. (9). $[A]$ and $[B]$ are independent of the eigenvalue $\lambda$ and, for these standard problems, various methods for determining the eigenvalue are well populated; implemented as standard subroutines and sometimes given as blackbox. On the contrary, for Eq. (9), $[A]$ is function of $\lambda$ requesting different approach. And moreover, the series $[A_i]$ in Eq. (11) differs from one problem to the other, and consequently universal solver for Eq. (9) seems not to be developed.

In what follows, we will present a method of conversion of Eq. (9) into the standard-type like Eq. (12) or (13). Here, we do not take into account the right-hand side $[B]$ of Eq. (9)

$$[A(\lambda)][x] = \{0\}$$

(14)

Introduce the following vector series $\{x_i\}$:

$$\{x_i\} = l^i \{x\}$$

(15)

where

$$\{x_0\} = \{x\}$$

(16)

Thus

$$\{x_{i-1}\} = l i \{x_i\} \quad (i = 0, 1, \ldots, m-1)$$

(17)

From Eqs. (11) and (14),

$$[A_0]\{x_0\} + [A_1]\{x_1\} + \cdots + [A_m]\{x_m\} = \{0\}$$

(18)

Equations (17) and (18) are combined to express in a single
equation, as

\[
[A][x] = i[B][x]
\]  

where, using the N-th order unit matrix 1,

\[
[A] = \begin{bmatrix}
A_{m-1} & A_{m-2} & \cdots & A_1 & A_0 \\
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0
\end{bmatrix}
\]  

(19)

and

\[
[B] = \begin{bmatrix}
-A_m & 0 & \cdots & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\]

(20)

and

\[
[x] = [x_{m-1} \ x_{m-2} \ \cdots \ x_0]^T
\]  

(21)

Equation (19) is in the form of Eq. (13) and therefore can be treated by the existing standard subroutines. Matrices \([A]\) and \([B]\) are of order \(mN\), computed easily by \([A]\).

**NUMERICAL EXAMPLES**

We consider, as the first example, a rectangular region \(L_x = 0.9\) and \(L_y = 0.4\) under the boundary condition shown in Fig. 1. The rigorous analytical solution is known as
where, s and t are the mode numbers in the x and y directions respectively. Boundary element analysis, using the subroutine named HEQRVD in Nagoya University Computer Center, was performed using 26 and 44 constant boundary elements. The order m of the matrix \([A_i]\) was taken up to their magnitude less than \(10^{-12}\) (for the range \(k = 0 \sim 6, m = 12\)). The obtained results of the eigenvalue \((k)\) for \(k = 0 \sim 6\) are shown in Table 1. "Standard EVP" denotes the result obtained by the present standard formulation, and "LU/Newton" that by the LU decomposition and Newton iteration shown in [6, 7]; both are compared with the analytical results.

We consider, as the next example, the case of very close eigenvalues. A rectangular domain with rigid sides of their length \(L_x = 1, L_y = 0.99\) (Fig. 2) is almost square and its eigenvalues corresponding to the first eigenmodes in the x and y directions are \(\pi/L_x = 3.14\) and \(\pi/L_y = 3.17\). Results obtained by 32 constant elements (8 elements on each side) are \(k = 3.15\) and 3.18 respectively. Good agreement with the analytical solutions and further clear distinction of the nearly-spaced eigenvalues are found. For this problem, the previous iteration or direct search is not adequate; with coarse increments they are not distinguished and with fine increments efficiency becomes lower.

CONCLUDING REMARKS

The integral equation for the Helmholtz equation formulated by the MRM boundary element method was transformed into the standard eigenvalue problem, for which we can apply existing subroutines without assuming rough initial estimation. The method requiring only boundary discretization as main preprocessing is highly user-friendly.
REFERENCES


Table 1. Comparison of results.

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<th>LU/Newton</th>
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<td>26 elms.</td>
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Figure 1.

Figure 2.