Invited Paper
Boundary-domain element method applied to forced vibration of elastic plates with damping
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ABSTRACT
The paper is concerned with the boundary integral equation formulation for forced vibration of elastic plates and its numerical implementation. An approximate fundamental solution of static bending is used for this formulation. The resulting boundary integral equations are discretized by means of the boundary-domain element method. The forced vibration of elastic plates subjected to a time-harmonic exciting load is computed by a new computer code developed in this study, whereby the validity of the proposed method is demonstrated.

INTRODUCTION
Besides a number of analytical methods [1, 2], numerical methods of analysis have been widely applied to bending problems of elastic plates which are popularly used as structural components [3, 4]. Among them, the finite difference, finite element and boundary element methods can provide a powerful means for accurate, efficient analysis of bending problems of elastic plates [5-8].

In this study, the so-called direct integral equation formulation is presented for the forced bending vibration of elastic plates subjected to a time-harmonic exciting load. The exact fundamental solution of the steady-state bending vibration is expressed in terms of the modified Bessel function. Since its derivatives are included in the fundamental solutions concerning deflection
slope, bending moment and equivalent shearing force, their expressions are complicated and numerical computation should be carefully carried out. Furthermore, frequency is implicitly included in these fundamental solutions. This makes it difficult to compute the eigenfrequency and eigenmode by the boundary element method [9]. To circumvent these difficulties, we derive a set of integral equations using the fundamental solution of static bending problems, and regularize them up to an integrable order.

In the present formulation, all the integrals included can be evaluated by the standard Gaussian quadrature, and hence we can easily introduce higher order elements in numerical implementation of the formulation. The regularized integral equations are discretized by means of the boundary-domain element method [5], in which the boundary is divided into boundary elements and also the inner domain into domain elements. Through this discretization, we can obtain the linear system of simultaneous equations with respect to nodal unknowns on the boundary and in the inner domain. It is interesting to note that frequency is included in the system matrix in an explicit manner, and hence the standard solution procedure available for algebraic eigenvalue problems can be applied to the eigenfrequency analysis. Although the multiple reciprocity boundary element methods [10, 11] and other boundary-only formulations can be applied, the present solution procedure could provide a powerful means for structural-acoustic coupling problems and other coupled vibration problems. It should be pointed out that the proposed method of solution can be applied with equal ease to analysis of forced vibration with viscous damping. A few examples are computed by using the computer code newly developed in this study, whereby the validity of the proposed method is demonstrated.

INTEGRAL EQUATIONS AND SOLUTION SCHEME

We shall consider a homogeneous, isotropic, thin elastic plate of uniform thickness subjected to a lateral dynamic load. The governing equation of forced bending vibration with viscous damping can be written as follows [1-3, 6]:

\[ D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} = p \]  

where \( w(x,t) \) is the lateral deflection at a point \( x \) and time \( t \), \( \rho \) the mass density, \( h \) the uniform plate thickness, \( c \) the viscous damping coefficient, and \( p(x,t) \) the lateral load per unit area acting vertically to the plate surface, respectively. In equation (1), \( \nabla^4 \) is the biharmonic differential operator, whereas \( D \) is the flexural rigidity which is related with Young’s modulus \( E \), Poisson’s ratio \( \nu \) and
plate thickness \( h \) as follows:

\[
D = \frac{Eh^3}{12(1-\nu^2)}
\]  

(2)

In the present work, we consider the steady-state bending vibration under time-harmonic exciting forces. If the lateral load \( p \) is a time-harmonic exciting force with constant angular frequency \( \omega \), i.e. \( p(x,t) = P(x)e^{i\omega t} \), the deflection \( w \) can be expressed as \( w(x,t) = W(x)e^{i\omega t} \). Substituting \( p(x,t) \) and \( w(x,t) \) into equation (1), the governing equation can be expressed in terms of \( P(x) \) and \( W(x) \) as follows:

\[
DV^4W(x) - (\omega^2 \rho h - i\omega\sigma)W(x) = P(x)
\]  

(3)

Now, in the present integral equation formulation, we employ as an approximate fundamental solution the fundamental solution of static bending. Let \( W^* \) denote this approximate fundamental solution which satisfies the differential equation in an infinite domain:

\[
DV^4W^* = \delta(x-y)
\]  

(4)

where \( y \) is a source point and \( \delta(x-y) \) is the Dirac delta function. As well known, \( W^* \) is given by

\[
W^*(x,y) = \frac{1}{8\pi D} r^2 \ln r, \quad r = |x-y|
\]  

(5)

Using the above approximate fundamental solution, we can express the integral equation which relates the deflection at a point in the inner domain of plate to quantities on the plate boundary as follows [7, 8]:

\[
W(y) = \int_{\Gamma} \left[ W^*(x,y)V_n(x) - T_n^*(x,y)M_n(x) 
+ M_n^*(x,y)T_n(x) - V_n^*(x,y)W(x) \right] d\Gamma_x
+ (\omega^2 \rho h - i\omega\sigma) \int_{\Omega} W^*(x,y)W(x) d\Omega_x
+ \int_{\Omega} W^*(x,y)P(x) d\Omega_x
- \sum_{k=1}^{K_c} \left[ W^*(x,y)M_{nt}(x) \right]
+ \sum_{k=1}^{K_c} \left[ M_{nt}^*(x,y)W(x) \right], \quad y \in \Omega
\]  

(6)

where \( \Gamma \) is the boundary and \( \Omega \) the inner domain, respectively, whereas \( \sum \left[ \right] \) denotes the summation of the jump of a variable inside double brackets up to the total number of corner points \( K_c \). Furthermore, \( T_n = \partial w/\partial y \), \( M_n \), \( M_{nt} \) and \( V_n \)
are the outward normal derivative of deflection on the boundary, the bending moment, the twisting moment and the equivalent shear force, respectively, which are related to the deflection \( W \) by

\[
\begin{align*}
T_n &= \frac{\partial W}{\partial n} = W_i n_i \\
M_n &= -D \nu W_{ii} - D(1 - \nu) W_{ij} n_i n_j \\
M_{nt} &= -D(1 - \nu) W_{ij} n_i t_j \\
V_n &= Q_n + \frac{\partial M_{nt}}{\partial s} \\
Q_n &= -D W_{ij} n_i
\end{align*}
\]

(7)

where \( n_i \) and \( t_i \) are the unit normal vector and the unit tangent vector of the boundary, respectively, whereas \( s \) is the arc length along the boundary measured from a certain point on the boundary. There is the relation \( \partial(\ )/\partial s = \partial(\ )/\partial t \).

The fundamental solutions \( T_n^*, M_n^*, M_{nt}^* \) and \( V_n^* \) are related to \( W^* \) via equation (7); Their detailed expressions can be given as follows:

\[
\begin{align*}
T_n^* &= \frac{\partial W^*}{\partial n} = \frac{1}{8\pi D} r(2 \ln r + 1) \frac{\partial r}{\partial n} \\
M_n^* &= \frac{1 + \nu}{4\pi} (\ln r + 1) - \frac{1 - \nu}{8\pi} \left\{ 2\left( \frac{\partial r}{\partial n} \right)^2 - 1 \right\} \\
M_{nt}^* &= \frac{1 - \nu}{4\pi} \frac{\partial r}{\partial t} \frac{\partial r}{\partial n} \\
V_n^* &= Q_n^* + \frac{\partial M_{nt}^*}{\partial s} \\
Q_n^* &= M_{ij,j} n_i = -\frac{1}{2\pi r} \frac{\partial r}{\partial n} \\
\frac{\partial M_{nt}^*}{\partial s} &= \frac{1 - \nu}{4\pi r} \left( \kappa r - \frac{\partial r}{\partial n} \right) \left\{ 2\left( \frac{\partial r}{\partial n} \right)^2 - 1 \right\}
\end{align*}
\]

(8)

where \( \kappa \) is the curvature of the boundary at point \( x \).

Taking the limit \( y \in \Omega \rightarrow y \in \Gamma \) in equation (6), we obtain the integral equation concerning the deflection on the boundary. Since the order of singularity of \( V_n^* \) is \( 1/r \), the integral for \( V_n^* \) must be evaluated in the sense of the Cauchy principal value. We must consider four quantities as \( W, T_n, M_n \) and \( V_n \).
on the boundary except for the corner points and two of them are given as the boundary conditions. Therefore, two unknowns exist at a point on the smooth boundary and another equation is needed to get the solution. For this purpose, we take the outward normal derivative of equation (6). The resulting integral equation is used together with equation (6) for the solution. The integral equation thus obtained includes the hypersingular integral kernel. It is known that such an integral can be evaluated in the sense of the Cauchy principal value [7]. However, it is not an easy matter to evaluate numerically this integral when higher order elements are employed in the implementation. To circumvent this difficulty, we try to regularize the integrals up to an integrable order, and then differentiate it.

First, we regularize equation (6) by the same procedure as showed previously by the authors [7] for the boundary element analysis of the static bending problem, that is,

\[
\int_{\Gamma} \left[ W^*V_n - T_n^* M_n + M_n^* T_n - Q_n^* \{ W - W(y) \} + M_{nt}^* \frac{\partial W}{\partial t} \right] d\Gamma
\]

\[+ (\omega^2 \rho h - i\omega c) \int_{\Omega} W^* W d\Omega + \int_{\Omega} W^* P d\Omega - \sum_{k=1}^{K} [ W^* M_{nt} ] = 0 \]  

(9)

All the integrals in equation (9) can be evaluated in the standard sense.

Differentiating equation (9) with respect to \( y_i (i = 1,2) \) and multiplying both sides by \( n_i(y) \), and regularizing it up to an integrable order by means of the same procedure as showed for the static bending analysis, we can obtain

\[
\int_{\Gamma} \left[ \tilde{W}^* V_n - \tilde{T}_n^* M_n + \tilde{M}_n^* \{ T_n - W_k(y)n_k \} \right]
\]

\[- \tilde{Q}_n^* \{ W - W(y) - r_k W_k(y) \} + \tilde{M}_{nt}^* \left\{ \frac{\partial W}{\partial t} - W_k(y)n_k \right\} \right] d\Gamma
\]

\[+ (\omega^2 \rho h - i\omega c) \int_{\Omega} \tilde{W}^* W d\Omega + \int_{\Omega} \tilde{W}^* P d\Omega - \sum_{k=1}^{K} [ \tilde{W}^* M_{nt} ] = 0 \]  

(10)

where \( (\tilde{\cdot}) = \partial(\cdot)/\partial y_i \cdot n_i(y) \). All the integrals in equation (10) can also be evaluated in the standard sense.

The domain integrals in equations (9) and (10) include the unknowns of the nodal deflection in the inner domain. Hence, we must set up equation (6), the integral equation including the source point \( y \) in the inner domain, to the total
number of nodal points of domain elements. In equation (6), the integral for \( V_n^* \) has the order of the Cauchy principal value as \( y \) approaches \( x_0 \) on the boundary. Hence, in order to calculate accurately the nodal deflection near the boundary, it is recommended to rewrite equation (6) in a similar manner to the derivation of equation (9) as follows:

\[
W(y) = \int_\Gamma \left[ W^* V_n - T_n^* M_n + M_n^* T_n - Q_n^* \left\{ W - W(x_0) \right\} + M_{nt}^* \frac{\partial W}{\partial t} \right] d\Gamma \\
+ W(x_0) + (\omega^2 \rho h - i\omega c) \int_\Omega W^* W d\Omega + \int_\Omega W^* P d\Omega - \sum_{k=1}^{K_c} \left[ W^* M_{nt} \right]
\]

(11)

where \( W(x_0) \) is the deflection of \( x_0 (\in \Gamma) \) nearest to the source point \( y (\in \Omega) \). Although the order of singularity of \( Q_n^* \) is \( 1/r \), this singularity has been weakened in equation (11).

Now, we discretize equations (9), (10) and (11) by the boundary-domain element method [5]. In this method, the boundary is discretized into the boundary elements, whereas the inner domain is also discretized into the so-called domain elements. Using this solution scheme, we obtain the linear system of simultaneous algebraic equations concerning the nodal unknowns on the boundary and the nodal deflections in the inner domain.

First, discretizing equations (9) and (10) by the boundary-domain element method and taking the source point at each node on the boundary, we obtain

\[
[A] [X] + [C] [W^i] = [B] [Y] + [D]
\]

(12)

where \( \{ X \} \) and \( \{ Y \} \) denote the unknown vector and the prescribed vector of the boundary nodal values, respectively, whereas \( \{ W^i \} \) denotes the unknown nodal deflection vector in the inner domain. Here we denote by \( [A] \), \( [B] \) and \( [C] \) the coefficient matrices which can be computed using the fundamental solutions, and \( \{ D \} \) denotes the known vector corresponding to the exciting load.

Discretizing equation (11) in a similar manner, and taking the source point at each node in the inner domain, we obtain

\[
[I] [W^i] + [a] [X] + [c] [W^i] = [b] [Y] + [d]
\]

(13)

where \( [I] \) denotes the unit matrix. We also denotes by \( [a] \), \( [b] \) and \( [c] \) the coefficient matrices and by \( \{ d \} \) the known vector.

Combining equations (12) and (13), we can finally arrive at the following
system of simultaneous algebraic equations:

\[
\begin{bmatrix}
A & C \\
I + c & I
\end{bmatrix}
\begin{bmatrix}
X \\
W^i
\end{bmatrix} =
\begin{bmatrix}
B \\
b
\end{bmatrix}
\begin{bmatrix}
Y \\
d
\end{bmatrix}
\]  \hspace{1cm} (14)

Solving equation (14) with respect to \{X\} and \{W^i\}, we can obtain all the unknowns on the boundary and the nodal deflections in the inner domain.

**NUMERICAL EXAMPLES**

In order to demonstrate the validity of the proposed method of solution, a few examples are analyzed using the new computer code developed in this study.

As the first example, we shall consider a circular plate with radius \(R = 1\)m and thickness \(h = 0.01\)m as shown in Figure 1. The whole boundary contour is either clamped or simply-supported, and the circular plate is subjected to the uniform time-harmonic exciting load of amplitude \(P = 100\)Pa. The material constants are assumed as shown in Figure 1. Frequency dependence of the deflection is investigated in such a way that the dimensionless angular frequency \(\lambda = R(\omega^2\rho h/D)^{1/4}\) is changed from 0.0 to 10.0. Using quadratic isoparametric elements for the boundary-domain element mesh, we use the following three kinds of discretization:

i) 12 boundary elements, 36 domain elements, 109 nodal points,
ii) 16 boundary elements, 64 domain elements, 193 nodal points,

![Circular plate subjected to uniform time-harmonic load](image)

**Figure 1.** Circular plate subjected to uniform time-harmonic load

Characteristics of the plate:

- \(E = 2.0 \times 10^{11}\)Pa
- \(\nu = 0.3\)
- \(\rho = 7.8 \times 10^3\)kg/m\(^3\)

Clamped or simply-supported
iii) 20 boundary elements, 100 domain elements, 301 nodal points.

A typical boundary-domain element mesh is shown in Figure 2 for the case iii).

First, we show the computational results in case of no damping. Figures 3 and 4 illustrate the relations between the dimensionless angular frequency $\lambda$ and the dimensionless central point deflection amplitude $|W_0|/(PR^4/D)$. Comparison is made with the analytical solutions [12] for the clamped plate and the

![Figure 2. Boundary-domain element mesh using quadratic isoparametric elements (in case of 20 boundary elements)](image)

![Figure 3. Central point deflection amplitude of clamped circular plate without damping](image)
simply-supported one, respectively. It can be seen that the present results are accurate even for 16 boundary elements and more accurate when 20 boundary elements are used. Tables 1 and 2 show the first three eigenfrequencies of symmetrical eigenmodes, and also the errors between the present results and the analytical solutions [13] for the clamped plate and the simply-supported one, respectively, when 20 boundary elements are used. The symbol $\lambda_i^n$ denotes the eigenfrequency for the $i$-th symmetrical eigenmode. The present results are also in good agreement up to higher frequencies. Figures 5 and 6 show the displacements in radial direction compared with the analytical solutions [12] near the eigenfrequencies for the clamped plate and the simply-supported one, respectively, in case of 20 boundary elements. The deflection is made dimensionless by using the magnitude of central point deflection amplitude $|W_0|$. The results obtained are in good agreement with the analytical solutions.

Next, we present the computational results obtained assuming viscous damping. Figures 7 and 8 show the relations between the dimensionless angular frequency $\lambda$ and the dimensionless central point deflection amplitude $|W_0|/(PR^4/D)$. Comparison is made with the analytical solutions [12] for the clamped circular plate and the simply-supported one, respectively, in case of 20 boundary elements. Three different damping ratios, i.e. $\zeta = 0.001$, 0.1 and 0.5 are taken into consideration, where $\zeta = c/c_c$, $c_c = 2\rho h\omega_1$ and $c_c$ denotes the critical damping coefficient. It can be seen that the present results are accurate even if viscous damping is considered.
Table 1. First three eigenfrequencies for symmetrical eigenmodes of clamped circular plate (in case of 20 boundary elements)

<table>
<thead>
<tr>
<th></th>
<th>$\lambda_f$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>3.196</td>
<td>6.306</td>
<td>9.439</td>
</tr>
<tr>
<td>Present</td>
<td>3.196</td>
<td>6.308</td>
<td>9.477</td>
</tr>
<tr>
<td>Error %</td>
<td>0.000</td>
<td>0.032</td>
<td>0.403</td>
</tr>
</tbody>
</table>

Table 2. First three eigenfrequencies for symmetrical eigenmodes of simply-supported circular plate (in case of 20 boundary elements)

<table>
<thead>
<tr>
<th></th>
<th>$\lambda_f$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical</td>
<td>2.222</td>
<td>5.452</td>
<td>8.611</td>
</tr>
<tr>
<td>Present</td>
<td>2.222</td>
<td>5.451</td>
<td>8.632</td>
</tr>
<tr>
<td>Error %</td>
<td>0.000</td>
<td>0.018</td>
<td>0.244</td>
</tr>
</tbody>
</table>

Figure 5. Displacement amplitude in radial direction of clamped circular plate without damping (in case of 20 boundary elements)
Figure 6. Displacement amplitude in radial direction of simply-supported circular plate without damping (in case of 20 boundary elements)

Figure 7. Central point deflection amplitude of clamped circular plate with viscous damping (in case of 20 boundary elements)
Figure 8. Central point deflection amplitude of simply-supported circular plate with viscous damping (in case of 20 boundary elements)

Next, we shall consider a square plate with side length $L = 1\text{m}$ and thickness $h = 0.01\text{m}$ as shown in Figure 9. Two opposite sides are free and the other sides are clamped or simply-supported, and the square plate is subjected to the uniform time-harmonic exciting load of amplitude $P = 100\text{Pa}$. The material constants are assumed as shown in Figure 9. Frequency dependence of the deflection is investigated as the dimensionless angular frequency $\lambda = L(\omega^2ph/D)^{1/4}$ is changed from 0.0 to 12.0. Computation is carried out for the following two discretizations using quadratic isoparametric elements:
i) 16 boundary elements, 16 domain elements, 65 nodal points,
ii) 32 boundary elements, 64 domain elements, 225 nodal points.

A typical boundary-domain element mesh is shown in Figure 10 for the case ii).

In this example, no damping is taken into consideration. Figures 11 and 12 show the relations between the dimensionless angular frequency $\lambda$ and the dimensionless central point deflection amplitude $|W_0|/(PL^4/D)$. Comparison is made with the analytical solutions [12] for the two cases, i.e. two opposite sides are clamped and simply-supported. It can be seen that accuracy of computations...
Figure 12. Central point deflection amplitude of square plate with two opposite sides simply-supported

decreases in the higher frequency range in case of 16 boundary elements, but the present results can be accurate even in the high frequency range if 32 boundary elements are used. Figures 13 and 14 show the results obtained using 32 boundary elements for the displacements in the direction of $x_1$-axis, where $|W_0|$ is the magnitude of central point deflection. Comparison is made with the analytical solutions [12] near the eigenfrequencies for the two boundary conditions. Both the results in these two figures are in good agreement with the analytical solutions.

It is interesting to point out that the proposed method of solution is applicable with equal ease to arbitrary cases of plate geometry and boundary conditions considering the effect of viscous damping. Furthermore, the finer the boundary-domain element mesh is, the better the numerical results are.

CONCLUSION

This paper has presented an integral equation formulation of the steady-state forced bending vibration of elastic plates with viscous damping in which the fundamental solution of static bending is used as an approximate fundamental solution. The resulting integral equations have been so regularized that no Cauchy principal value or no hypersingular integrals are included. A new computer code was developed and applied to several examples to demonstrate the validity of the proposed method of solution.
Figure 13. Displacement amplitude in $x_1$-axis direction of square plate with two opposite sides clamped (in case of 32 boundary elements)

Figure 14. Displacement amplitude in $x_1$-axis direction of square plate with two opposite sides simply-supported (in case of 32 boundary elements)
Parts of this work were financially supported by Grant-in-Aid for Scientific Research of Japan’s Ministry of Education, Science and Culture. The authors express their cordial thanks for this financial support.

REFERENCES

13. p.170 of Ref.(2)