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Keynote Address

A history of boundary elements

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ABSTRACT:

Boundary element methods have played an important role in the numerical solution of many engineering problems. This overview attempts to place BEM in context with other boundary methods and with it's own "before— and after—computer" history.

INTRODUCTION:

This is a personal view of the historical development of boundary element methods (BEM) and their relationship to boundary methods in general. My introduction to them was as a PhD student and Guggenhein Fellow at the Guggenheim Institute for Flight Structures at Columbia University in the mid-1950's. The director, Prof. H. H. Bleich, suggested to a new faculty member, Prof. M. B. Friedman, that he and I (his first PhD student) look at acoustic shock scattering by underwater structures by replacing the body by some saltus problem, i.e. surfaces of sources and sinks. There were no boundary elements per se at that time, but in this group of mixed backgrounds, there was familiarity with Green's functions, sources and doublets, Huygens' principle, etc., e.g. Morse and Feshbach[1]. I came from an applied math background with a minor in aeronautical engineering and was surrounded by some of the best faculty in solid and fluid mechanics, Prof.'s Mindlin, Boley, Herrmann, Freudenthal, Weiner, Salvadori, Bleich, Arnold and Friedman. I have long felt that this mixture of

fluid and solid mechanics gave me a definite historical advantage. There also was no finite element method to try to emulate — what we did was in its own context, building on boundary methods in general.

I feel the importance of recognizing the long history of boundary methods in general and the existence, even today, of alternative points of view to what we now call BEM. It will be assumed that this audience has some familiarity with BEM, but not necessarily its relatives. Underlying everything must remain the reason for having such methods in the first place, i.e. there are physically based, engineering problems which require some sort of solution, at least through mathematical models which represent their salient physical behavior. To understand where we are requires understanding of where we came from in all things, computational mechanics included.

Since there are "volume" oriented methods of finite differences and finite elements, as competitors to boundary methods, one question is how is a boundary method distinguished from a volume method ? The view taken here is that a boundary method allows the solution at individual interior field points directly from boundary values without requiring any intermediate interior point calculations. Anything else will be considered a volume method and not part of this discussion, although definitely part of the overall repertoire of solution methods.

The first step in considering boundary methods themselves is to distinguish between formulations, i.e. alternative ways of describing a problem, and computational methods for solving those formulations. In considering computational methods, there are "numerical methods", which generally require some discretization of the geometrical domain, and "approximate methods", which generally require some trunctation of an infinite series solution, both of which arise here.

BOUNDARY INTEGRAL EQUATIONS AND RELATED FORMULATIONS:

The classical boundary integral equation formulation (BIE), for which the BEM is the most frequently used solution technique, is usually understood as based either mathematically on Green's theorem and/or physically on surfaces of source and doublet distributions. Consider the problem, in physical terms, to be defined by a set of source points which produce a solution at a set of field points. If the source points and the field points are both located on that surface on

which the boundary conditions are given, and if the relationship between these two sets of data is in the form of an integral or integro-differential equation, as in Green's theorem, we would call this a boundary integral equation (BIE) formulation. If the integral equation is then solved by a discretization of the boundary surface into elements leading to a set of linear algebraic equations on unknown nodal values (or mean values, Shaw[2]), we could call this a BEM solution technique. However, there exist methods which do not place the source and the field points on the same surface and that there are methods of solution which do not involve discretization of the boundary surface into elements. For example, if the source points are placed on the given boundary but the field points are placed outside of the original domain, where Green's theorem requires a zero solution or "null-field", an integral equation on the unknown boundary values results. This is the classical null-field formulation, to be solved by either elements or eigenfunction expansions. If field points are placed both inside and outside the original domain, on simple geometrical surfaces, and the resulting Green's integrals solved for example by eigenfunction expansions appropriate to the governing equation and the simple field point geometries, the unknown boundary values may be eliminated giving the field point solution directly in terms of the given boundary values. This is the "T-matrix" method, used widely in wave scattering and radiation problems, e.g. Varadan and Varadan[3], give an overview of this method based on the original work of Waterman[4]. If the field points are placed on the given boundary surface but the source points are placed on simple geometrical surfaces outside of the original domain, these are the "embedding formulations" which may be solved by either elements or by eigenfunction expansions, i.e. "embedding solution methods", e.g. Shaw and Huang[5]. This last approach actually includes the classical Trefftz method when appropriately interpreted, Shaw, Huang and Zhao[6]. These are all still boundary methods based on "source" distributions and integral equations. Some of these predate what we call BEM and others were devloped concurrently with BEM. The whole idea of source/doublet distributions goes back well beyond what we generally consider to be the beginnings of our standard BEM approaches. Indeed, Huygen's principle, developed in the 17th century and described by Baker and

Copson[7] contains many ideas that relate to modern BEM. Even a historical development divided into before--computer (B.C.) and after—computer (A.C.) contains an amazing overlap of boundary methods.

B.C; BEFORE—COMPUTER YEARS:

The earliest formation and solution of physical problems governed by partial differential equations were clearly boundary methods. The classical separation of variables method is based on an orthogonal eigenfunction expansion, specific to the governing equation, the geometry and the boundary conditions given, whose coefficients are determined succesively and directly from the boundary conditions. The classical Green's function approach gave the solution for all field points directly in terms of a quadrature of boundary values, using "the" Green's function specific to the governing equation, the geometry and the boundary conditions given, to eliminate any unknown boundary values. In both cases, interior solutions could be found directly in terms of that specific interior point and the boundary values and do not involve any other interior calculations.

While finite difference methods, which are clearly a volume approach, were the dominant numerical technique during these B.C. years, some numerical and/or approximate boundary methods were still present. Consider the classical Trefftz method which involves an eigenfunction expansion that satisfies the governing equations but does not form an orthogonal basis set for the the given boundary geometry. The coefficients of this expansion are found without orthogonality of the basis set on the given geometry, thus requiring the solution of a system of linear algebraic equations of order equal to the number of terms used in the expansion. We can note here that this method is equivalent to the embedding integral eigenfunction method. This embedding integral approach was also used in its element form by Munk[8] and by von Karman[9]. Some early BEM solutions are also B.C., e.g. those by Prager[10] and by Massonnet[11]. In fact, the first half of my own PhD thesis in 1960 on transient acoustic wave scattering and subsequently my first BEM paper, Friedman and Shaw[12] was carried out on a Monroe desk calculator. Since the second half, Shaw and Friedman[13], was done on an IBM 704, this thesis might be 0 C.

Boundary integral equation methods (BIE) were also discussed well B.C., e.g. Proudman[14], who regretfully dismissed as requiring too much computation to be of practical use AT THAT TIME.

A.C.: AFTER—COMPUTER YEARS

It is obvious that the advent of wide access to the digital computer has led to an explosion of approximate/numerical methods for the solution of engineering problems. This is as true for FEM as for BEM, Since there are a number of books on BEM, beginning with the first proceedings of Cruse and Rizzo[15], the first text by Jaswon and Symm[16] and the first teaching text of Brebbia[17], with many more to come, this era will be discussed primarily in the context of particular techniques. However, it is clear in retrospect that the time was right for this approach and several groups were independently working on it including the McDonnell-Douglas group under Smith and Hess, e.g. Hess[18] and Hess and Smith[19] and the English group under Jaswon[20], on potential theory, several U.S. Navy labs, e.g. with Chen and Schweikert[21] , Chertock[22], Schenck[23], etc. on acoustics and the Rizzo and Cruse team on solids, e.g. Cruse and Rizzo[24]. Much of the earliest work was done on problems in infinite domains and was days called the surface integral method, the boundary integral method, etc. The phrase BEM was not coined until the meeting organized by Tom Cruse and Frank Rizzo in 1975 where it was discussed as a solution method for the BIE rather than the basis of the method itself. It seemed more useful to keep the formulation nature clear than naming a technique to sound like FEM.

SOME SPECIFIC BOUNDARY INTEGRAL FORMULATIONS:

There have been applications of BEM to a variety of physical problems and corresponding equation types. Numerical solution of hyperbolic systems often lead to uncoupled systems of linear algebraic equations due to the finite wave speeds involved — in fact, a prominent BEM researcher once told me that my first papers on transient acoustic scattering were not proper BEM studies since I made use of this "time retardation" to uncouple my algebraic equations and thus solve no more than an occasional 2 X 2 system at a sharp corner. I convinced him that this was BEM, but considering the state of

computers at that time, a "kinder" BEM. At my PhD defense in 1959, I was asked how I would solve a time harmonic wave scattering problem and I replied that I would start from a rest state and build up to a steady state as the limit of a hyperbolic system — we were so afraid of large, e.g. 20 X 20 linear algebraic systems, at that time that this was considered a reasonable response. Four years later, Banaugh and Goldsmith[24] presented the time harmonic scattering problem, an elliptic system, using a computer throughout thus illustrating the rapid changes going on in computational power. Of some historical interest was the fact that Banaugh was in the audience at the Fourth U.S. Congress of Applied Mechanics when I presented my 1962 paper and asked if we had considered time harmonic cases, probably seeing his current PhD thesis being "scooped". I replied, to his relief, only as a long time solution to a hyperbolic system. Parabolic systems also have a definite place in the study of diffusion problems, but it is in the elliptic realm that BEM has really flourished and this will be the basis of the rest of this discussion. This is especially true in view of the fact that both hyperbolic and parabolic systems are routinely transformed, typically by Laplace or Fourier transforms, into equivalent elliptic systems in a transform space, solved there and then inverted back to the time domain. Consider then problems governed by elliptic partial differential equations as the primary basis for discussion. The domain under consideration will be referred to as D and the boundary to this domain as δD. D may be finite or infinite, but δD is assumed to be finite unless stated otherwise. The two classical elliptic equations considered here will be the Laplace equation for potential problems and the Helmholtz equation for time harmonic wave equations. The treatment for elastostatics or time-harmonic elastodynamics follows much the same pattern as will be discussed for these examples and will only be explicitly considered when substantial differences occur. Let us begin with either a Laplace equation or a Helmholtz equation under Dirichlet boundary conditions for simplicity,

$$\nabla^2 u(\vec{r}) = 0 \text{ in } D ; \; u(\vec{r}) = f(\vec{r}) \text{ on } \delta D \qquad [1\text{-a}]$$

$$\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = 0 \text{ in } D ; \; u(\vec{r}) = f(\vec{r}) \text{ on } \delta D \qquad [1\text{-b}]$$

If this is a separable geometry for this equation and boundary

condition, classical separation of variables will work. If the problem is recast into an integral equation by Green's theorem,

$$c\ u(\vec{r}) = \int_{\delta D} \{ G(\vec{r}, \vec{r}_0)\ \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\ \partial G(\vec{r}, \vec{r}_0)/\partial n_0 \}\ dS(\vec{r}_0) \quad [2]$$

where $u(\vec{r}_0)$ is replaced by $f(\vec{r}_0)$, $c = (0, 1/2, 1)$ for the field point, $\vec{r}$, lying ouside, on (a smooth part) or inside of δD when viewed from D, and $\hat{n}$ is the outward normal to δD from D. This integral is taken in a Cauchy principal value sense for the middle case. The major difference in formulation between eq. [1-a] and [1-b] is the Green's function used. Since u is known on δD, if G is chosen to be zero on δD — i.e. "the" Green's function — this equation is reduced to a quadrature of known values for any interior field point, i.e. $c = 1$,

$$u(\vec{r}) = \int_{\delta D} - f(\vec{r}_0)\ \partial G(\vec{r}, \vec{r}_0)/\partial n_0\ \ dS(\vec{r}_0) \quad [3]$$

If "the" Green's function is not known, "a" Green's function, usually but not necessarily the infinite space point source solution, may be used. With $\vec{r}$ on δD, this leads

$$u(\vec{r})\ /\ 2 = \int_{\delta D} \{ G(\vec{r}, \vec{r}_0)\ \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\ \partial G(\vec{r}, \vec{r}_0)/\partial n_0 \}\ dS(\vec{r}_0) \quad [4]$$

which is the standard direct BIE, involving only boundary values of the unknowns. This equation may be interpreted as having layers of sources, G, and doublets, $\partial G/\partial n$, of strengths $\partial u/\partial n$ and u respectively. In this sense, the solution $u(\vec{r})$ varies from the correct interior solution for $\vec{r}$ just inside of δD to zero just outside of δD with an "average" value of (1/2) u(r) used for $\vec{r}$ on δD. Again, for a Dirichlet problem, $u(\vec{r}_0)$ is known on δD making this an integral equation of the first kind on $\partial u/\partial n$. It could be converted to an integral equation of the second kind by differentiating eq. [2] in the outward normal direction to D at that prospective field point before allowing $\vec{r}$ to actually approach δD and then going through the limiting process, e.g. Shaw and Friedman[12], resulting in

$$(\partial u(\vec{r})/\partial n)/2 = \int_{\delta D} \{\partial G(\vec{r},\vec{r}_0)/\partial n\ \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\ \partial^2 G(\vec{r},\vec{r}_0)/\partial n_0 \partial n\} dS(\vec{r}_0) \quad [5]$$

where the unknown now appears outside of as well as within the integral. Similar results are found for Neumann and mixed boundary condition problems. Before examining solution methods for these forms, let us consider some "variations on this theme". If $\vec{r}$ is placed outside of δD, c = 0 and we have a "null—field" formulation, e.g.

$$0 = \int_{\delta D} \{ G(\vec{r}, \vec{r}_0)\, \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\, \partial G(\vec{r}, \vec{r}_0)/\partial n_0 \}\, dS(\vec{r}_0) \quad [6]$$

which also involves only boundary values. If instead of using the mathematical form of Green's theorem, and its corresponding physical interpretation in terms of layers of sources and doublets, we went directly to such layers of arbitrary strength, $\sigma(\vec{r})$ on δD, we would have the indirect formulations, e.g.

$$u(\vec{r}) = \int_{\delta D} \sigma(\vec{r}_0)\, G(\vec{r}, \vec{r}_0)\, dS(\vec{r}_0) \quad [7]$$

with the normal derivative, $\partial u(\vec{r})/\partial n$, on δD given by

$$\partial u(\vec{r})/\partial n_{-/+} = (+/-)(1/2)\sigma(\vec{r}) + \int_S \sigma(\vec{r}_0)\, \partial[G(\vec{r},\vec{r}_0)]/\partial n_0\, dS(\vec{r}_0) \quad [8]$$

where the + and — refer to an approach to the boundary from inside and outside of D respectively. Eq. [7] allows σ to be found by placing $\vec{r}$ on δD and using the given boundary condition. Eq. [7] and eq. [8] then determine u and $\partial u/\partial n$ anywhere in terms of these boundary source strengths.

The T matrix and embedding equation methods are in a sense complementary methods both of which avoid the difficulty of principal value integrals. The T-matrix method uses the standard boundary integral form of eq. [2] but for solution points $\vec{r}_i$ interior to D and $\vec{r}_e$ exterior to D, i.e. as

$$u(\vec{r}_i) = \int_{\delta D} \{ G(\vec{r}_i, \vec{r}_0)\, \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\, \partial G(\vec{r}_i, \vec{r}_0)/\partial n_0 \}\, dS(\vec{r}_0) \quad [9\text{-a}]$$

$$0 = \int_{\delta D} \{ G(\vec{r}_e, \vec{r}_0)\, \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\, \partial G(\vec{r}_e, \vec{r}_0)/\partial n_0 \}\, dS(\vec{r}_0) \quad [9\text{-b}]$$

Eq. [9-b] allows information about $\partial u/\partial n$ on δD to be related to the given Dirichlet boundary conditions on u, typically in the form of

coefficients of an eigenvalue expansion, while eq. [9-a] incorporates that information to yield a relationship between u at some interior field point and the given boundary condition. The actual values of $\partial u/\partial n$ on δD need never be determined explicitly.

The embedding integral formulation reverses locations of source and solution points and is more closely related to the indirect approach of eqs. [7] and [8]. Here the "sources" of strength $\sigma(\vec{r})$ are placed on a convenient surface exterior to the domain D and the solution points are taken on δD. The exterior embedding surface, S_e, is usually taken to be a simple geometric form, e.g. a circle in 2D or a sphere in 3D. The governing integral equation is then

$$u(\vec{r}) = \int_{S_e} \sigma(\vec{r}_0)\, G(\vec{r}, \vec{r}_0) \quad dS(\vec{r}_0) \qquad [10]$$

with $\vec{r}$ on the original boundary δD. This integral form can then be solved for $\sigma(\vec{r}_0)$. Once $\sigma(\vec{r}_0)$ is known, the solution point may be placed anywhere in D to determine $u(\vec{r})$ there. Derivatives of u may be found by differentiating eq. [10] since only $G(\vec{r}, \vec{r}_0)$ is affected by derivatives in the $\vec{r}$ coordinate system. Clearly doublets or combinations of sources and doublets of varying strengths could be used qually well.

SOME BOUNDARY SOLUTION METHODS:

The above section dealt with boundary formulations. The actual numbers will come out of solution techniques which are categorized into "numerical", typically element methods, and "approximate", typically eigenfunction expansion methods. Both of these approaches have been used successfully, even on the same formulations although it does appear that certain formulations have come to be thought of in terms of one specific approach. Thus, the boundary integral formulation of eq. [4] is inevitably tied to an element solution technique while the T-matrix formulation of eqs. [9-a] and [9-b] is always solved by eigenfunction expansions. This is fine as long as the restriction to one approach is based on careful consideration of other approaches which have then been found lacking in some respect or other and not due to lack of knowledge of alternatives. A number of simple illustrations are given in an appendix, but actual numbers are left to the literature. It is clear that there are unanswered questions

as to why certain solution methods do not appear to have been tried with some of these formulations. One goal then of future BIE study should be to investigate why these apparent omissions have occured, but some statements may be made in general even now. Eigenfunction expansions require some expansion of the fundamental Green's function in an appropriate set of basis functions, i.e. the eigenfunctions for some geometry not necessarily that given in the problem but typically of some similar shape, size, or other property. Such formulations have two distinct forms, one for $|\vec{r}| > |\vec{r}_0|$ and another for $|\vec{r}| < |\vec{r}_0|$. Such distinctions may be made when $\vec{r}$ lies on one surface completely within or without of the surface containing $\vec{r}_0$ as is the case for null-field methods, T-matrix methods and embedding methods, all of which have eigenfunction expansion solution procedures. However, for boundary integral formulations where $|\vec{r}|$ may be both larger than and less than $|\vec{r}_0|$ depending on their relative locations on the same surface, this Green's function expansion may not be divided into such distinct forms. This does not mean that this approach can not be used for BIE but helps explain why it has not been used in practice - in fact, I am aware of only one attempt to use it, Sharma[26], and even this was not a direct expansion. Nevertheless, this topic is currently being studied for one very good reason. As BEM codes go from two to three dimensions, the number of elements and thus the size of the influence coefficient matrices increases drastically. However the calculation of each coefficient is only slightly more difficult in the 3D case than in the 2D case, i.e. the standard Gauss coefficients are known either way. The same problem by eigenfunction expansions would have 3D basis functions but probably not too many more terms than in the 2D eigenfunction expansion, i.e. the number of unknown expansion coefficients would not increase dramatically. However, the calculation for each term in the corresponding matrix relating these expansion coefficients would certainly have increased substantially. Thus while the assembly time for either solution approach might be comparable, the storage required is far less for the eigenfunction approach. Although this might appear to be a minor problem since memory is becoming virtually unlimited, the number of elements being used in ever increasingly difficult problems is also increasing. In order to have solution codes that do not require suoercomputer capabilities, it

might pay to consider this distinction. Nevertheless, it appears that BEM will stand as the "method of choice" for general applications since it is far more flexible in its application to a variety of problems than ia the T-matrix which requires specific eigenfunction choices for each case.

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APPENDIX: SOLUTION METHODS

I) ELEMENT METHODS:

a) Classical BEM: Here eq. [4] is solved numerically by dividing the boundary δD into M elements and assuming some functional variation of the dependent variable over each element. The simplest approach uses a 'constant' shape function for this variation (which may be better viewed as a mean value, e.g. Shaw[2]. The field point, $\vec{r}$, is taken successively at some representative location, e.g. the midpoint, of each element providing M equations on the M unknown dependent values.

$$u_i/2 = \sum_{k=1}^{M} \int_{\delta D_k} \{G(\vec{r}_i, \vec{r}_0)\, \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\, \partial G(\vec{r}_i, \vec{r}_0)/\partial n_0\} dS(\vec{r}_0)$$

$$= \sum_{k=1}^{M} (\partial u/\partial n)_k \int_{\delta D_k} G(\vec{r}_i, \vec{r}_0)\, dS(\vec{r}_0) - u_k \int_{\delta D_k} \partial G(\vec{r}_i, \vec{r}_0)/\partial n_0\, dS(\vec{r}_0)$$

$$= \sum_{k=1}^{M} (\partial u/\partial n)_k\, G_{ik} - u_k\, \hat{H}_{ik} \quad ; \quad i = 1, \ldots M \qquad [A\text{-}1]$$

For a Dirichlet problem, u_i is known as f_i ; the solution for $(\partial u/\partial n)_k$ is

$$(\partial u/\partial n)_k = [\, G_{ik} \,]^{-1} [\, 0.5\, \delta_{ij} + \hat{H}_{ij} \,]\, f_j \qquad [A\text{-}2]$$

Other boundary conditions lead to other forms of essentially this same result. This is a collocation method leading to an M × M system of linear algebraic equations; other forms of solution are also possible, e.g. over-determined systems solved by least-mean-square error methods.

b) Null - field Methods: Here the field or solution point is placed at M locations outside of D as in eq. [6]. The same discretization as above leads to

$$0 = \sum_{k=1}^{M} \int_{\delta D_k} \{\, G(\vec{r}_i, \vec{r}_0)\, \partial u(\vec{r}_0)/\partial n_0 - u(\vec{r}_0)\, \partial G(\vec{r}_i, \vec{r}_0)/\partial n_0 \,\} dS(\vec{r}_0)$$

$$= \sum_{k=1}^{M} (\partial u/\partial n)_k \int_{\delta D_k} G(\vec{r}_i, \vec{r}_0)\, dS(\vec{r}_0) - u_k \int_{\delta D_k} \partial G(\vec{r}_i, \vec{r}_0)/\partial n_0\, dS(\vec{r}_0)$$

$$= \sum_{k=1}^{M} (\partial u/\partial n)_k\, G_{ik} - u_k\, \hat{H}_{ik} \quad ; \quad i = 1, \ldots M \qquad [A\text{-}3]$$

which again represents a system of M equations on M unknowns. The coefficients G_{ik} and $\hat{H}_{ik}$ are obtained from non-singular integrals; the resulting algebraic system is not as robust as BEM, undoubtedly due to the lack of the strong diagonal term provided by the singularity contribution.

c) Indirect BEM: Here an intermediate variable is used, in this case representing the strength of a source layer, $\sigma(\vec{r})$. The same boundary discretization is used leading to

$$u(\vec{r}_i)=u_i=\sum_{k=1}^{M}\int_{\delta D_k}\sigma(\vec{r}_0)\,G(\vec{r}_i,\vec{r}_0)\,dS(\vec{r}_0)=\sum_{k=1}^{M}\sigma_k\,G_{ik}\;;\;i=1,.\,M \quad [A\text{-}4]$$

For a Dirichlet problem, this may be solved for the values σ_k. Then values of $u(\vec{r})$ in the field may be found by eq. [7] with $\vec{r}$ off of the boundary. The jump in normal derivative at the boundary by

$$(\partial u(\vec{r})/\partial n_{-/+})_i=(+/-)(1/2)\,\sigma_i+\sum_{k=1}^{M}\sigma_k\,\hat{H}_{ik} \quad [A\text{-}5]$$

d) Embedding Element Method: The approximation of eq. [10] by an element approach involves subdividing S_e into segments. In 2D, S_e could be a circle for convenience if the original geometry is roughly of equal dimensions in all directions, although other embedding geometries could be used for more 'distorted' goemetries. Then these subdivisions are simply circular arcs on which integrations may be readily performed without geometrical shape function approximations. Obviously, the same advantage holds in three dimensions as well. Taking N segments, of equal size unless there is some clear reason to do otherwise, and average values of σ,the segmentation of eq. (2) yields

$$u(\vec{r}_i)=\sum_{k=1}^{M}\sigma_k\int_{S_{ek}}G(\vec{r}_i,\vec{r}_e)\,dS(\vec{r}_e)=\sum_{k=1}^{M}\sigma_k\,G_{ik}\;;\;i=1,\ldots,M \quad [A\text{-}6]$$

i.e. a system of M equations on the M unknowns, solved by collocation. The coefficients G_{ik} are calculated from non-singular integrals.

II) EIGENFUNCTION EXPANSION METHODS:

a) T-matrix Methods and Null-field Methods: The T-matrix method is the standard eigenfunction expansion method comparable in

significance to the BEM for element methods. It is used primarily on Helmholtz exterior wave scattering and radiation problems, but could be applied to other problems in essentially the same format. The variables u and $\partial u/\partial n$ on the boundary as well as the free space Green's function, G, and the field values for u are expanded in a basis set of linearlly independent eigenfunctions which satsify the governing equation although not necessarily the appropriate boundary conditions on the given geometry. An origin within the surface δD is used for both interior and exterior problems; an exterior radiation problem will be considered here. A surface, S_a, is drawn outside of D, e.g. for a circle in 2D or a sphere in 3D this could simply be $|\vec{r}_a| = a$ where a is always less than the value of r_0 to the boundary, S_0, from some central origin. Then these expansions are

$$u(\vec{r}_0) = \sum_{k=0}^{\infty} \alpha_k \ \hat{\psi}_k(\vec{r}_0) \ ; \ \vec{r}_0 \in \delta D \qquad [A\text{-}7]$$

$$\partial u(\vec{r}_0)/\partial n_0 = \sum_{k=0}^{\infty} \beta_k \ \hat{\psi}_k(\vec{r}_0) \ ; \ \vec{r}_0 \in \delta D \qquad [A\text{-}8]$$

$$G(\vec{r}_a, \vec{r}_0) = \sum_{m=0}^{\infty} \gamma_m \ \psi_m(\vec{r}_a) \ \hat{\psi}_m(\vec{r}_0) \ ; \ |\vec{r}_a| < |\vec{r}_0| \qquad [A\text{-}9]$$

and eq. [9-b] becomes

$$0 = \sum_{m=0}^{\infty} \gamma_m \psi_m(\vec{r}_a) \{ \sum_{k=0}^{\infty} \beta_k \int_{\delta D} \hat{\psi}_k(\vec{r}_0) \hat{\psi}_m(\vec{r}_0) dS_0 - \alpha_k \int_{\delta D} \hat{\psi}_k(\vec{r}_0) \partial \hat{\psi}_m(\vec{r}_0)/\partial n_0 dS_0 \} \qquad [A\text{-}10]$$

Since the ψ_m are linearly independent, this provides a relationship between the coefficients α and β,

$$\sum_{k=0}^{\infty} \beta_k \ P_{km} = \sum_{k=0}^{\infty} \alpha_k \ Q_{km} \qquad [A\text{-}11]$$

which can be solved for α given β, β given α, etc. At this point, this is really the null-field method. However, the T-matrix method goes one step further. Taking a second surface within D, e.g. a circle or sphere or radius b such that $b > |\vec{r}_0|$, the field at this surface can also be expanded in this set of basis functions except that the form of the Green's function expansion changes slighty.

$$u(\vec{r}_b) = \sum_{m=0}^{\infty} \hat{\alpha}_m \ \hat{\psi}_m(\vec{r}_b) \ ; \ \vec{r}_b \in D \qquad [A\text{-}12]$$

$$G(\vec{r}_b, \vec{r}_0) = \sum_{k=0}^{\infty} \gamma_k \ \psi_k(\vec{r}_0) \ \hat{\psi}_k(\vec{r}_b) \ ; \ | \vec{r}_0 | < | \vec{r}_b | \qquad [A\text{-}13]$$

Then eq. [9-a] becomes

$$\sum_{m=0}^{\infty} \hat{\alpha}_m \ \hat{\psi}_m(\vec{r}_b) = \sum_{m=0}^{\infty} \gamma_m \ \hat{\psi}_m(\vec{r}_b) \ \{ \sum_{k=0}^{\infty} \beta_k \int_{\delta D} \psi_k(\vec{r}_0) \ \hat{\psi}_m(\vec{r}_0) \ dS_0 - \sum_{k=0}^{\infty} \alpha_k \int_{\delta D} \hat{\psi}_m(\vec{r}_0) \ \partial\psi_k(\vec{r}_0)/\partial n_0 \ dS_0 \ \} \qquad [A\text{-}14]$$

where the summation index has been rearranged for convenience from the first set of expansions. The integrals are similar to those of the first set, with modified functions. However, since the $\hat{\psi}_m$ are again linearly independent, this reduces to

$$\hat{\alpha}_m = \gamma_m \{\beta_k \ P^*_{km} - \alpha_k \ Q^*_{km}\} = \gamma_m \ \{[P_{kt}]^{-1} \ Q_{qt} \ P^*_{km} - \delta_{kq} \ Q^*_{km}\}\alpha_q \qquad [A\text{-}15]$$

with no sum over m and sums over the other indices indicated by repeated values. This relates the field values of u to the boundary values of u directly in this Dirichlet case; other forms of boundary condition would follow this same procedure.

b) Embedding Methods: Here the sources are placed on an 'embedding' surface exterior to D, e.g. on a circle in 2D or a sphere in 3D (although other geometries could also be used). All functions are again expanded in terms of a set of linearly independent basis functions which satisfy the governing equation. In this case however, these are chosen to also be orthogonal over the embedding surface. Then for an exterior problem, the embedding surface will have $| \vec{r}_e | < | \vec{r}_0 |$ and eq. [10] becomes

$$u(\vec{r}_0) = \sum_{k=0}^{\infty} \alpha_k \hat{\psi}_k(\vec{r}_0) = \int_{S_e} \sum_{m=0}^{\infty} \sigma_m \ \psi_m(\vec{r}_e) \sum_{k=0}^{\infty} \gamma_k \ \hat{\psi}_k(\vec{r}_0) \ \psi_k(\vec{r}_e) \ dS(\vec{r}_e) \qquad [A\text{-}16]$$

or since the $\hat{\psi}$ are linearly independent and $u(\vec{r}_0)$ is known,

$$\alpha_k = \sum_{m=0}^{\infty} \sigma_m \ \gamma_k \int_{S_e} \psi_m(\vec{r}_e) \ \psi_k(\vec{r}_e) \ dS(\vec{r}_e) = \sum_{m=0}^{\infty} \sigma_m \ \gamma_k \ R_{km} \qquad [A\text{-}17]$$

which can be solved for σ_m and thus give $u(\vec{r})$ everywhere as an expansion in terms of $\hat{\psi}_k$; this is essentially the Trefftz method.