Solution of torsional problems by BEM

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Abstract

Practical engineering problems in torsion, heat conduction,and fluid flow are governed by the Poisson-type equation. The objective of this study is to reduce the Poisson equation to the Laplace equation by substituting a particular solution so that the problem can be handled by the practical boundary element analysis. Care should be taken in these cases to transform the boundary conditions accordingly.

This paper presents the application of B.E.M. to two torsional problems with different shapes of cross sections and the numerical results are compared with the analytical solutions.

Introduction

Most engineering problems which are expressed in a differential form can only be solved in an approximate manner due to their complexity. The irregular boundaries of practical problems preclude any analytical solution of the governing equations and numerical methods become the only feasible means of obtaining adequately precise and detailed results. The finite element method is now the most widely used numerical method for the solution of large boundary value problem.

The boundary element method has emerged as a powerful alternative to the finite element method. BEM reduces the dimensionality of the basic process by one; i.e., for two-dimensional problems the analysis generates a one-dimensional integral equation and for three-dimensional problems only two-dimensional surface integral equations arise.

BEM involves modeling only the boundary geometry of the system. Once the necessary boundary information has been obtained, values of the solution variables can then be calculated at any subsequently selected interior points.

Torsion

The semi-inverse method proposed by Saint-Venant is used to solve the problem of non-circular prismatic bars subjected to torque. In each particular case the

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solution is reduced to the determination of the stress function satisfying the differential equation and the boundary condition. The solution of the torsion problem is thus reduced in finding a function u(x,y) such that

$$\frac{\partial}{\partial x}\left(\frac{1}{G}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{G}\frac{\partial u}{\partial y}\right)=-2\theta \qquad \textit{in the domain} \tag{1}$$

$$u = 0 \qquad \textit{on the boundary} \tag{2}$$

In Equation (1) G is the shear modulus of elasticity and θ is the torsion angle per unit length. The problem variable u(x,y) is the stress function (Prandtl's stress function).

u must be constant along the boundary of the cross section. In the case of singly connected boundaries (solid bars), this constant can be chosen arbitrarily. Equation (2) shows that such constant is assumed to be zero.

When a prismatic bar is composed of a single material, Equation (1) can be nondimensionalized and written in the following form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-2 \tag{3}$$

Equation (3) is Poisson's equation which can be reduced to Laplace's equation if we let $u = u^* - (x^2 + y^2)/2$. The problem is then reduced to solving

$$\frac{\partial^2 u^*}{\partial x^2}+\frac{\partial^2 u^*}{\partial y^2}=0 \tag{4}$$

with the boundary condition $u^* = (x^2 + y^2)/2$ when $u = 0$ at the boundary.

The Problem of a Bar with a Rectangular Cross Section

Consider a bar with a rectangular cross section, with its center at the origin, and with sides 2a and 2b (Figure 1).

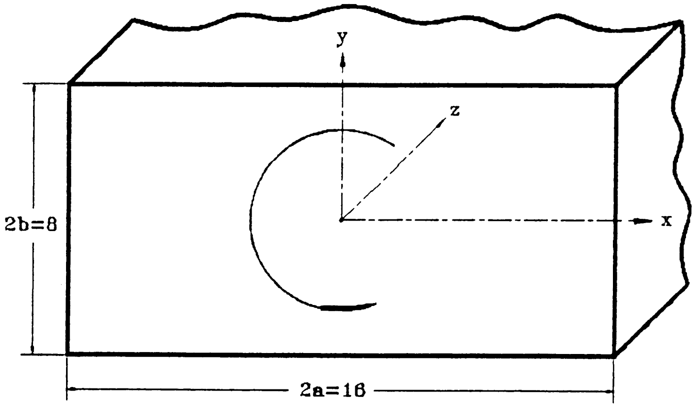

Figure 1. A bar with a rectangular cross section

Lebedev, Skalskaya, and Ulfyans (1965) performed the study using the Fourier method and obtained the following analytical solution [2]

$$u(x,y) = (x+a)(a-x) - 32\frac{a^2}{\pi^3}\sum_{n=0}^{\infty}\left[\frac{\sin[(2n+1)\pi(x+a)/2a]}{(2n+1)^3}\,\frac{\cosh[(2n+1)\pi y/2a]}{\cosh[(2n+1)\pi b/2a]}\right] \qquad (5)$$

(-a <= x <= a, -b <= y <= b)

Boundary Element Model

From symmetry, only upper right quarter of the cross section needs to be discretized. The bar was discretized using 12 quadratic boundary elements and has 21 internal nodes. Figure 2 shows the boundary element discretization. The nodes of a quadratic boundary element are at the ends and in the middle of the element. The stress function at the boundary is calculated with the boundary condition $(x^2+y^2)/2$.

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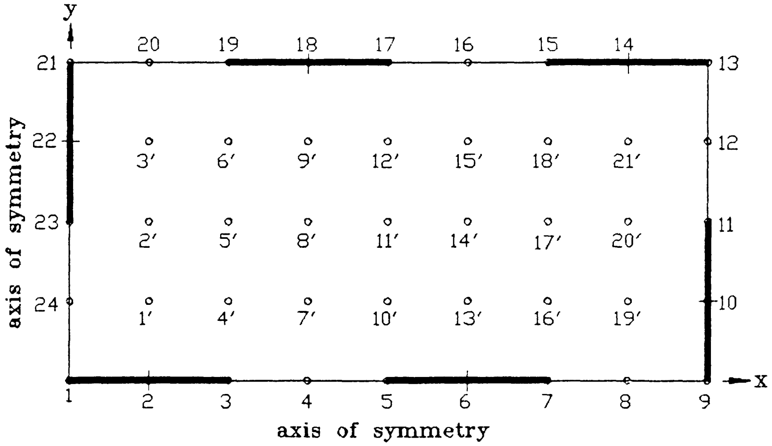

Figure 2. Discretization of upper right quarter of cross section into 12 boundary elements

Figure 3 shows the boundary element solution.

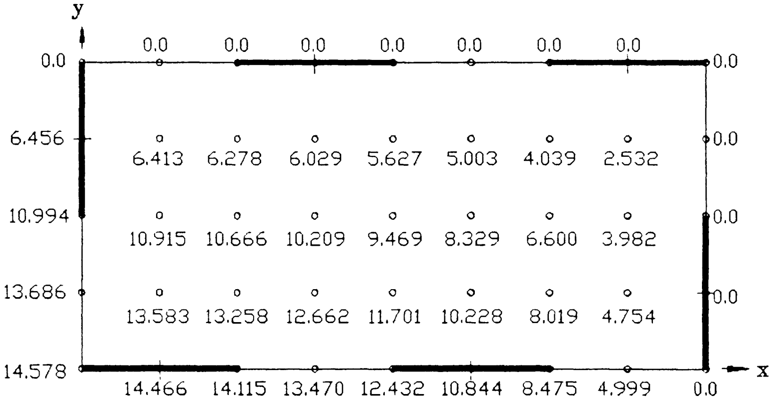

Figure 3. Stress function values

Results

The numerical results for stress functions at both boundary and internal nodes are summarized in Table I,where they are compared with the analytical solution [Equation (5)].

Table I

x	y	BEM	analytical	% error
Boundary Nodes				
0.0	0.0	14.578	14.576	0.01 %
1.0	0.0	14.466	14.464	0.01 %
2.0	0.0	14.115	14.114	0.01 %
3.0	0.0	13.470	13.469	0.01 %
4.0	0.0	12.432	12.431	0.01 %
5.0	0.0	10.844	10.843	0.01 %
6.0	0.0	8.475	8.473	0.02 %
7.0	0.0	4.999	4.998	0.02 %
0.0	3.0	6.456	6.455	0.02 %
0.0	2.0	10.994	10.993	0.01 %
0.0	1.0	13.686	13.684	0.01 %
Internal Nodes				
1.0	1.0	13.583	13.581	0.01 %
1.0	2.0	10.915	10.914	0.01 %
1.0	3.0	6.413	6.412	0.02 %
2.0	1.0	13.258	13.257	0.01 %
2.0	2.0	10.666	10.665	0.01 %
2.0	3.0	6.278	6.278	0.00 %
3.0	1.0	12.662	12.661	0.01 %
3.0	2.0	10.209	10.208	0.01 %
3.0	3.0	6.029	6.029	0.00 %
4.0	1.0	11.701	11.700	0.01 %
4.0	2.0	9.469	9.469	0.00 %
4.0	3.0	5.627	5.627	0.00 %
5.0	1.0	10.228	10.227	0.01 %
5.0	2.0	8.329	8.329	0.00 %
5.0	3.0	5.003	5.003	0.00 %
6.0	1.0	8.019	8.018	0.01 %
6.0	2.0	6.600	6.600	0.00 %
6.0	3.0	4.039	4.042	0.07 %
7.0	1.0	4.754	4.753	0.02 %

7.0	2.0	3.982	3.980	0.05 %
7.0	3.0	2.532	2.532	0.00 %

The Problem of a Circular Shaft Weakened by a Radial Crack Going from the Surface of the Shaft to its Axis

Consider a circular shaft with a radius of unity weakened by a radial crack going from the surface of the shaft to its axis (Figure 4). The crack is approximated as a gap with 0.01 clearance.

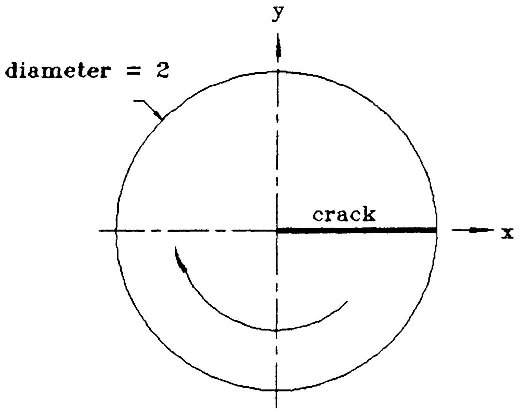

Figure 4. A circular shaft with a radial crack

Lebedev,Skalskaya,and Ulfyans (1965) performed the study using the eigenfunction method and obtained the following analytical solution [2].

$$u(r,\phi)=32\frac{a^2}{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{r}{a}\right)^{(2n+1)/2}-\left(\frac{r}{a}\right)^2}{(2n+1)\left[16-(2n+1)^2\right]}\sin\frac{(2n+1)\phi}{2} \qquad (6)$$

where a is the radius of the shaft.

Boundary Element Model

From symmetry, only upper half of the cross section needs to be discretized. The shaft was discretized using 23 quadratic boundary elements and has 20 internal nodes. Figure 5 shows the boundary element discretization. Again the stress function at the boundary is calculated with the boundary condition $(x^2+y^2)/2$.

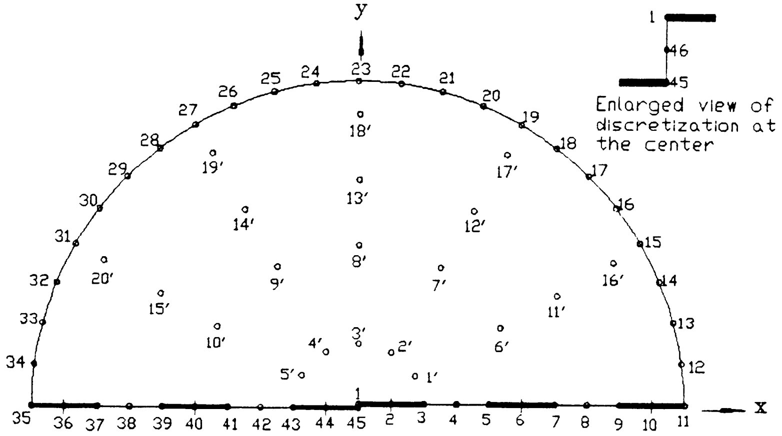

Figure 5. Discretization of half of cross section into 23 boundary elements

Figure 6 shows the boundary element solution.

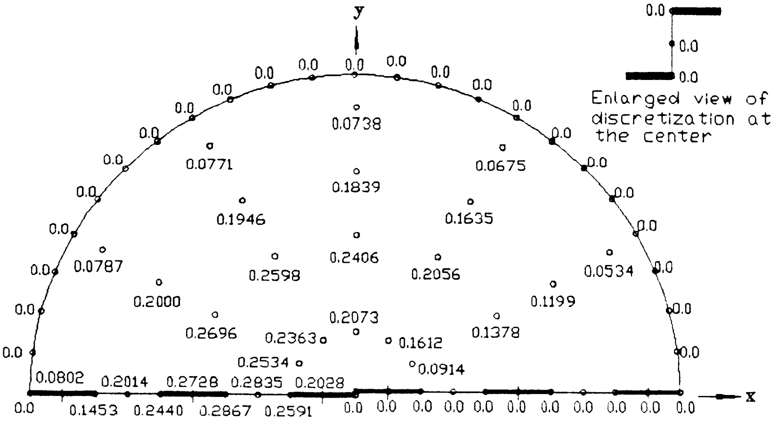

Figure 6. Stress function values

Results

The numerical results for stress functions at both boundary and internal nodes are summarized in Table II, where they are compared with the analytical solution [Equation (6)].

Table II

x	y	BEM	analytical	% error
Boundary Nodes				
-0.9000	0.0000	0.0802	0.0791	1.39 %
-0.8000	0.0000	0.1453	0.1463	0.68 %
-0.7000	0.0000	0.2014	0.2013	0.05 %
-0.6000	0.0000	0.2440	0.2435	0.16 %
-0.5000	0.0000	0.2728	0.2719	0.33 %
-0.4000	0.0000	0.2867	0.2854	0.46 %
-0.3000	0.0000	0.2835	0.2817	0.64 %
-0.2000	0.0000	0.2591	0.2564	1.01 %
-0.1000	0.0000	0.2028	0.1987	2.01 %
Internal Nodes				
0.1732	0.1000	0.0914	0.0952	3.99 %
0.4330	0.2500	0.1378	0.1401	1.64 %
0.6062	0.3500	0.1199	0.1212	1.07 %
0.7794	0.4500	0.0534	0.0538	0.74 %
0.1000	0.1732	0.1612	0.1633	1.29 %
0.2500	0.4330	0.2056	0.2068	0.58 %
0.3500	0.6062	0.1635	0.1642	0.43 %
0.4500	0.7794	0.0675	0.0677	0.30 %
0.0000	0.2000	0.2073	0.2084	0.53 %
0.0000	0.5000	0.2406	0.2412	0.25 %
0.0000	0.7000	0.1839	0.1842	0.16 %
0.0000	0.9000	0.0738	0.0739	0.14 %
-0.1000	0.1732	0.2363	0.2364	0.04 %
-0.2500	0.4330	0.2598	0.2597	0.04 %
-0.3500	0.6062	0.1946	0.1946	0.00 %
-0.4500	0.7794	0.0771	0.0771	0.00 %
-0.1732	0.1000	0.2534	0.2516	0.72 %
-0.4330	0.2500	0.2696	0.2691	0.19 %
-0.6062	0.3500	0.2000	0.1998	0.10 %
-0.7749	0.4500	0.0787	0.0786	0.13 %

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Conclusion

The boundary element method was validated by applications to two torsional problems. The results of this study once again show that BEM provides high accuracy of solution and greatly simplifies modeling. BEM involves modeling only the boundary, or surface, of the system. Once the boundary information has been obtained,values of the physical quantities can then be calculated at any internal nodes.

The computations have been performed in single precision with Prime 6150 and the BEM results have been obtained by using the computer code for potential problems described in [1].

This study also shows that the Poisson-type equation, when reduced to the Laplace equation by substituting a particular solution, can be handled by the simple and practical boundary element analysis. Care should be taken in these cases to transform the boundary conditions accordingly.

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