Local integral equations

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Abstract

A reliable computational technique is developed for the solution of two-dimensional (2-d) transient heat conduction problems in anisotropic media with continuously variable material coefficients. The meshless point interpolation is employed for the approximation of the spatial variation of the temperature field or its Laplace-transform. The coupling amongst the nodal values of the approximated field is given by integral equations considered on local sub-domains. Three kinds of local integral equations are derived from physical principles instead of using a weak-form formulation. The accuracy and the convergence of the proposed techniques are tested by several examples and compared with exact benchmark solutions.

Keywords: integral equation methods, fundamental solutions, integral balance equation, meshless interpolation, Laplace transform, time stepping technique, heat conduction, functionally graded materials (FGMs), anisotropy

1 Introduction

The recent progress in the development and research of functionally graded materials (FGMs) enhanced also the interest in the development of numerical methods for the solution of boundary or initial-boundary value problems in non-homogeneous continuous media. In FGMs, the composition and the volume fraction of the FGMs constituents vary gradually, giving a non-uniform microstructure with continuously graded macroproperties. Owing to the composite structure, the material properties are directionally dependent or anisotropic in general. The solution of transient heat conduction problems in anisotropic and non-homogeneous media is a complex task in general, even in the case of material linearity. In FEM formulations, the variational principles are
applied to the whole structure, hence the geometry of the analyzed domain is to be modeled. This is the principal restriction preventing to develop truly meshless formulations utilizing the same variational principles as in FEM, because a background mesh is needed at least for the required integrations over the whole structure. In pure boundary integral formulations, the relevant unknowns are localized only on the boundary of the analyzed domain. Although the BEMs belong to mesh reduction techniques, mesh generation is still required either owing to the modeling of the geometry of the global boundary or to the approximation of the relevant boundary quantities. Moreover, pure boundary integral formulations are possible only if the fundamental solution of the governing differential operator is known. Basically, the fundamental solutions are available in closed forms and can be expressed in terms of elementary functions only for linear differential operators with constant coefficients corresponding to homogeneous and continuous media. For some special classes of material coefficients variations, the governing differential operators with variable coefficients can be converted to simple operators with constant coefficients by using variable transformations [1, 2]. Such an approach is very elegant but not general.

In this paper, we present our recent development on the general numerical methods for transient heat conduction analysis in anisotropic and continuously non-homogeneous media. As regards the time variable, we employ both the commonly used time-stepping scheme and the Laplace-transform (LT) technique. The spatial variation of the field variables is approximated by using the meshless point interpolation method (PIM) utilizing both the polynomial and the radial basis functions. The use of a domain-type approximation is sufficient to take the boundary conditions into the consideration. For coupling among the nodal values of the field variable we derive three different kinds of integral equations which are considered on local sub-domains around interior nodes. The integral equations are derived from physical principles as an equivalent of the governing differential equation instead of using various weak-form formulations of the governing differential equation. Since the size, the shape and the composition of sub-domains are arbitrary, the resulting system matrix can be arranged to be sparse. Moreover, having chosen a simple geometrical shape for sub-domains in the meshless approach, the numerical integrations can be performed in a simple way. To improve the numerical stability of the methods for transient problems, a special procedure for selecting supporting nodes is proposed. The accuracy, convergence and reliability of the proposed methods have been verified by several numerical tests with using the exact benchmark solutions.

2 Governing equation and boundary value problems

The governing equation for transient heat conduction in anisotropic and continuously non-homogeneous media is given by the following partial differential equation with variable coefficients [3]
where \( u(x,t) \) is the temperature field and \( w(x,t) \) is the density of heat sources, \( \lambda_{ik}(x) \) is the tensor of thermal conductivity coefficients, while the specific heat per unit mass \( c(x) \) is a scalar quantity like the mass density \( \rho(x) \).

The first term on the left-hand-side of Eq. (1) is the divergence of the heat flux vector

\[
q_i(x,t) = -\lambda_{ik} (x) u_{,k}(x,t)
\]

and the second term is the rate of the temporal change of the volumetric heat density.

Eventually, the physically reasonable boundary conditions of the problem can be of the Dirichlet, Neumann and Robin types. The boundary conditions are to be supplemented by the initial condition, which in the present parabolic problem is the initial value of the temperature

\[
u(x) = u(x,0)
\]

3 Integral form of the balance equation (IFBE)

The general physical balance principles of a continuum theory take the form of integral equations. The governing equations or field equations take the form of differential equations which can be derived from these integral equations because they hold for all arbitrary but small material domains \( \Omega^c \). Following a reverse path, we can get the integral form of the balance principles by integrating the governing differential equations over \( \Omega^c \subset \Omega \). Thus, integration of Eq. (1) with subsequent application of the Gauss divergence theorem leads to

\[
\int_{\partial \Omega^c} q_i(\eta,t) n_i(\eta) d\Gamma(\eta) + \int_{\Omega^c} \rho(x)c(x) \frac{\partial u(x,t)}{\partial t} d\Omega(x) = \int_{\Omega^c} w(x,t) d\Omega(x),
\]

where \( n_i(\eta) \) denotes the Cartesian components of the unit outward normal vector at \( \eta \) on the boundary \( \partial \Omega^c \) of the sub-domain \( \Omega^c \). The integral equation (4) will be referred to as the integral form of the balance equation (IFBE).

4 Local integral equations (LIEs) with singular fundamental solution

Having used the singular fundamental solutions for the governing differential operator, one can derive an integral representation of the field variable which satisfies the governing differential equation exactly. Hence, the integral
equations based on singular fundamental solutions are physically admissible as an integral equivalence of the governing differential equation. Owing to the lack of a closed-form fundamental solution for the governing differential operator with generally variable coefficients, it is appropriate to rewrite Eq. (1) as

$$\lambda_{ij}^c u_{ij}(x,t) + \left( \tilde{\lambda}_{ij}(x)u_{ij}(x,t) \right)_i - \rho(x)c(x) \frac{\partial u(x,t)}{\partial t} = -w(x,t) \quad ,$$

(5)

where $\lambda_{ij}^c = \lambda_{ij}(x^c)$, and $\tilde{\lambda}_{ij}(x) = \lambda_{ij}(x) - \lambda_{ij}^c$ is a fluctuation of the thermal conductivity coefficients inside a sub-domain $\Omega^c$ surrounding the point $x^c$. For the sake of brevity, we shall assume $w(x,t) = 0$.

Let $G(x - y)$ be the fundamental solution of the stationary governing equation with constant coefficients in an infinite space, i.e.,

$$\lambda_{ij}^c G_{ij}(x - y^c) = -\delta(x - y^c) \quad .$$

(6)

According to Chang et al. [4], Eq. (6) has the following solution in 2-d case

$$G(x - y^c) = -\frac{1}{2\pi \sqrt{\Lambda^c}} \ln\left( \sqrt{R} \right) , \quad R = \left( \lambda^c \right)_{ij}^{-1} r_i r_j , \quad r_i = x_i - y_i^c \quad ,$$

(7)

where $|\Lambda^c|$ denotes the determinant of the matrix $\Lambda^c$ whose elements are given by $\lambda_{ij}^c$. In view of Eq. (1), we can immediately write the integral identity

$$\int_{\Omega^c} G(x - y^c) \left[ \left( \lambda_{ij}(x)u_{ij}(x,t) \right)_i - \rho(x)c(x) \frac{\partial u(x,t)}{\partial t} \right] d\Omega(x) = 0 \quad .$$

(8)

Making use of the Gauss divergence theorem, we arrive at the integral relationship

$$\int_{\partial\Omega^c} \lambda_{ij}(\eta)u_{ij}(\eta,t)n_i(\eta)G(\eta - y^c)d\Gamma(\eta) - \int_{\Omega^c} \lambda_{ij}(x)u_{ij}(x,t)G_{ij}(x - y^c)d\Omega(x) -$$

$$- \int_{\Omega^c} \rho(x)c(x) \frac{\partial u(x,t)}{\partial t} G(x - y^c)d\Omega(x) = 0 \quad ,$$

(9)

in which $y^c \in \Omega^c$. This integral equation will be referred to as the local integral equation (LIE) of the 1st kind.
Splitting $\lambda_{ij}$ as $\lambda_{ij} = \lambda_{ij}^c + \tilde{\lambda}_{ij}$ in the second integral term of Eq. (9), applying the Gauss divergence theorem to the term involving $\lambda_{ij}^c$, and invoking Eq. (6), we can obtain the following LIE of the 2nd kind

$$u(y^c, t) = \int_{\partial\Omega^c} \left[ \lambda_{ij}(\eta)u_{ij}(\eta, t)\eta_i(\eta)G(\eta - y^c) - u(\eta, t)\lambda_{ij}^c\eta_j(\eta)G_{ij}(\eta - y^c) \right] d\Gamma(\eta)$$

$$- \int_{\Omega^c} \tilde{\lambda}_{ij}(x)u_{ij}(x, t)G_{ij}(x - y^c) d\Omega(x) - \int_{\Omega^c} \rho(x)\frac{\partial u(x, t)}{\partial t} G(x - y^c) d\Omega(x). \quad (10)$$

Note that Eq. (10) is an integral representation of the temperature at an interior point $y^c$.

Although the integral equations (9) and (10) involve singular kernels, the integrands are finite and the integral equations are completely non-singular. This can be recognized from the fact that the singularity of gradients of the fundamental solution is cancelled by the Jacobian in the domain integrals, while the boundary integrals are non-singular due to $y^c \in \partial\Omega^c$.

When a domain-type approximation is employed for the temperature field, each of the integral equations (4), (9) and (10) is a physically meaningful constraint equation providing the proper interaction among the DOF, and can be used together with the collocated boundary conditions for the computation of the unknown DOF. The resulting equations are given by a system of ordinary differential equations for the time evolution of the nodal values of the temperature. Another possibility is to utilize the Laplace transform technique to avoid the solution of the ODEs, when the Laplace transforms of the nodal values are computed from the solution of a system of algebraic equations and the time variation is obtained by a numerical inversion of the transformation.

5 Meshless point interpolation method

In all meshless approximation techniques, shape functions have to be defined for the approximation of the field variable $u(x)$ (which is either the temperature field $u(x, t)$ or its Laplace transform $\bar{u}(x, p)$) within a sub-domain $\Omega_x$ using only nodes scattered arbitrarily in the analyzed domain without any predefined mesh to provide a connectivity of the nodes. Assuming a finite series representation of the field variable in a sub-domain $\Omega^q_x$ surrounding the nodal point $x^q$, the approximated field can be written as [5, 6]

$$u(x)|_{\Omega^q_x} = \sum_{k=1}^{N} R^{n(q,k)}(x)z^{(q,k)} + \sum_{k=1}^{M} P^{k}(x)\beta^{(q,k)}, \quad (11)$$
when using both the polynomial functions and the multiquadrics as the basis functions with \( n(q,k) \) being the global number of the \( k \)-th nearest nodal point of \( N \) supporting nodes corresponding to \( \mathbf{x}^q \). The radial basis is considered as multiquadrics

\[
R^n(x) = \left( |x - x^n|^2 + c^2 \right)^{m/2},
\]

with \( c \) and \( m \) being the shape parameters, and the polynomial basis \( P^k(x) \) is given by monomials. For \( k = 1, \ldots, 6 \) and \( M = 6 \) we have for instance

\[
P^k(x) \in \left\{ 1, x_1, x_1^2, x_1x_2, x_2^2 \right\}.
\]

According to our numerical experiences especially the stability tests [6], we propose to use \( N > M \). The choice \( M = 6 \) corresponds to the utilization of a complete quadratic polynomial basis. From (11), one can get the interpolation of the field variable in terms of its nodal values and the shape functions \( \varphi^{(q,j)}(x) \) as

\[
u(x)|_{\Omega_s^q} = \sum_{j=1}^{N} u(x^{n(q,j)})\varphi^{(q,j)}(x),
\]

where the shape functions are expressed via the basis functions [6].

Now, the boundary conditions can be easily implemented at boundary nodes, since the gradients of the field variable are approximated on \( \Omega_s^q \) in terms of its nodal values and gradients of the shape functions. Thus,

\[
u(\eta^b) = \tilde{u}(\eta^b) \text{ at } \eta^b \in \partial \Omega_D,
\]

\[
-n_i(\eta^b)\lambda_{ik}(\eta^b)\sum_{a=1}^{N} u(x^{n(h,a)})\varphi^{(h,a)}(\eta^b) = \tilde{q}(\eta^b) \text{ at } \eta^b \in \partial \Omega_N.
\]

As regards the numerical implementation of the integral equations on local sub-domains, it is very simple because the shape functions are given in the global Cartesian coordinate space. Moreover, we can use geometrically simple circular sub-domains around each interior node which is also the center of the sub-domain, since the size, the shape and the composition of sub-domains \( \Omega^c \) can be chosen arbitrarily. The radius of each circle is assumed to be equal to \( h \) which is the shortest distance of any two nodal points spread over the closed domain \( \Omega \cup \partial \Omega \).

The selection of supporting nodes, obeying the conditions \( n(q,1) = q \) and

\[
\left| x^{n(q,k+1)} - x^q \right| \geq \left| x^{n(q,k)} - x^q \right| \text{ for } k = 1, \ldots, N - 1,
\]

yields numerically stable
results [6] in stationary problems with implementation each of the IFBE and/or
the LIE of the 1st or 2nd kind. Nevertheless, certain instability has been observed
[7] with respect to the choice of the time steps in the θ-method, when the IFBE
in conjunction with the PIM is applied to transient problems. The accuracy of
this implementation is also very poor in the case of the LT-approach at very
early time instants for some nodal point distributions as pointed out in [7]. Both
these instabilities have been removed in the present paper by introducing a
perturbation into the selection of the supporting nodes. This perturbation consists
in the creation of a small gap in the distance between the first and the second
groups of the supporting nodes. The first group of the supporting nodes
\( \{ x^{n(q,k)} \}_{k=1}^{M} \) is given by the first \( M \) nearest nodes to the considered nodal point
\( x^q \), while the second group \( \{ x^{n(q,k)} \}_{k=M+1}^{N} \) is selected from the nodes whose
distance from \( x^q \) is greater than \( |x^{n(q,M)} - x^q| + g \), i.e. the nodes of the first and
the second groups are required to satisfy the following conditions
\( |x^{n(q,k+1)} - x^q| \geq |x^{n(q,k)} - x^q| \) for \( k = 1, \ldots, N-1 \) and \( |x^{n(q,k)} - x^{n(q,M)}| > g \) for
\( k = M + 1, \ldots, N \). With this modification, a pretty good stability can be achieved
with respect to the choice of \( g \).

6 Numerical examples

In order to test the proposed numerical methods, we consider examples for which
analytical solutions are available. The considered domain is a square \( L \times L \) with
the Dirichlet boundary conditions on both the bottom \( u(x_2 = 0) = u_0 \) and the top
\( u(x_2 = L) = u_L \) sides, while the Neumann conditions are assumed on the lateral
sides. The body source density is absent in \( \Omega \) and the material coefficients are
assumed as: \( \rho = 1 \), \( c(x) = c_0 f(x_2) \), \( \lambda_{ij}(x) = L_{ij} \lambda_0 f(x_2) \), with either \( L_{ij} = \delta_{ij} \)
in isotropic case, or \( L_{11} = L_{22} = 2 \), \( L_{12} = L_{21} = 1 \) in anisotropic case. The initial
condition is assumed to be constant \( v(x) = v_0 \) throughout the body.

We have used the shape parameter \( m = -2 \) for the radial basis function and
the polynomial basis is given by \( M = 6 \) monomials. The number of the radial
basis functions is \( N = 16 \), the shape parameter in multiquadrics is taken as
\( c = 2h \) [6], and the perturbation gap in supporting nodes is \( g = h / 2 \).
Furthermore, 10 LT parameters are used for each time instant in the LT-approach
together with the Stehfest’s inversion, and Galerkin’s implicit scheme with
\( \theta = 2 / 3 \) is employed in the time-stepping scheme. The difference between the
results by the LIEs of the 1st and the 2nd kind was extremely small in all
considered transient problems in non-homogeneous media.
The numerical results by the PIM-approach are compared also with the results obtained by implementation of the derived formulation using the standard quadrilateral quadratic elements (QQE-approach). For this purpose, we have employed a uniform distribution of nodal points. Fig. 1 shows the convergence of numerical results to exact values with increasing the density of nodal points in case of the anisotropic continuously non-homogeneous material with gradation according to the exponential law \( f(x) = \exp(\delta x_2 / L) \) with \( \delta = 3 \). The boundary and initial conditions have been specified as: \( u_0 = 1 \), \( u_L = 20 \), \( v_0 = u_0 \) and the used global average percentage error is defined by

\[
APE_t = 100 \left( \frac{1}{N_t} \sum_{a=1}^{N_t} \left[ u^e(x^a,t) - u^{ex}(x^a,t) \right]^2 \right) \left/ \frac{1}{N_t} \sum_{a=1}^{N_t} \left[ u^{ex}(x^a,t) \right]^2 \right.,
\]

where \( N_t \) is the total number of nodes on the closed domain \( \Omega \cup \partial \Omega \).

![Figure 1: Dependence of the accuracy at the time instant \( t = 0.5 \) on the nodal distance of the uniform node distribution by using either the LT-approach or the \( \theta \)-method.](image)

Lot of computations has been carried out for various material non-homogeneities including the exponential, power-law and trigonometric gradation of material coefficients. Fig. 2 is an illustration of the time evolution of the temperature at the square midpoint in case of trigonometric gradation:

\[
f(x) = \left[ \cos(\delta x_2 / L) + 5 \sin(\delta x_2 / L) \right]^2 \text{ with } \delta = 2 , \ u_0 = 0 , \ u_L = 100 , \ v_0 = u_0.\]
Figure 2: Time evolution of the temperature at the midpoint of the square with trigonometric material gradation by using the IFBE discretized with the QQE-interpolation and/or the PIM in the LT-approach.

7 Conclusions

In this paper, a comprehensive study on some recently developed numerical methods for the solution of 2-d transient heat conduction problems in anisotropic and continuously non-homogeneous materials is presented. No restrictions on the material non-homogeneities and anisotropy are assumed a-priori in the methods within the linear theory. The initial-BVP problem is solved either in the time-domain or in the Laplace-transformed domain in conjunction with an inverse algorithm. The following results can be drawn from the numerical experiments:

(i) Very satisfactory agreements of the numerical results by using either the QQE-interpolation or the meshless PIM with the exact results can be achieved for a lot of considered test examples, provided that a sufficient amount of nodal points is employed. The numerical calculations of the temperature gradients require a more dense distribution of the nodal points than the calculations of the temperature. Also, a large material gradation needs a rather high density of the nodal points. Nevertheless, the proposed numerical techniques have never been collapsed in all our numerical experiments.
(ii) The numerical stability of the meshless PIM has been extended also to transient problems by introducing a perturbation into the selection of the supporting nodes.

(iii) The meshless PIM requires much more CPU-times than the QQE-interpolation due to a much more complex evaluation of the shape functions. On the other hand, the preparation of input data is in the meshless PIM less time consuming. Another important advantage of the meshless PIM approach is for moving boundary value problems where a “remeshing” is required.

The presented methods can also be combined with other meshless approximations. The extension of the methods to three-dimensional problems and other engineering applications is straightforward.

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