

Meshfree direct and indirect local radial basis function collocation formulations for transport phenomena

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Abstract

This paper formulates an upgrade of the classical meshless Kansa method. It overcomes the principal large-scale bottleneck problem of this method. The formulation copes with the non-linear transport equation, applicable in solutions of a broad spectrum of mass, momentum, energy and species transfer problems. The domain and boundary of interest are divided into overlapping influence areas. On each of them, the fields (direct version) or second partial derivatives (indirect version) are represented by the multiquadrics radial basis function collocation on a related sub-set of nodes. Time-stepping is performed in an explicit way. The governing equation is solved in its strong form, i.e. no integrations are performed. The polygonisation is not present and the method is practically independent of the problem dimension. The complicated geometry is easy to cope with. The method is simple to learn and to code. The method can be straightforwardly extended to tackle other types of partial differential equations.

1 Introduction

Problems in science and engineering are usually reduced to a set of coupled partial differential equations. It is not easy to obtain their analytical solution, particularly in non-linear and complex-shaped cases, and discrete approximate methods have to be employed accordingly. The finite volume (FVM), the finite



element (FEM), and the boundary element methods (BEM) are most widely used among them at the present. Despite the powerful features of these methods, there are often substantial difficulties in applying them to realistic, geometrically complex three-dimensional transient situations with moving and/or deforming boundaries. A common complication in the mentioned methods is the need to create a polygonisation, either in the domain and/or on its boundary. This type of (re)meshing is often the most time consuming part of the solution process and is far from being fully automated. In recent years, a new class of methods is in development that do not require polygonisation but use only a set of nodes to approximate the solution. The rapid development of these types of meshless (polygon-free) methods and their classification is elaborated in the very recent monographs [1,2,3,4]. A broad class of meshfree methods in development today are based on Radial Basis Functions (RBFs) [5]. The RBF collocation method or Kansa method [6] is the simplest of them. This method has been further upgraded to symmetric collocation [7], to modified collocation [8] and to indirect collocation [9]. The method has been already used in a broad spectrum of computational fluid dynamics problems [10] such as the solution of Navier-Stokes equations [11] or porous media flow [12] and the solution of solid-liquid phase change problems [13]. In contrast to advantages over mesh generation, all the listed methods unfortunately fail to perform for large problems, because they produce fully populated matrices sensitive to the choice of the free parameters in RBFs. One of the possibilities for mitigating this problem is to employ the domain decomposition [14]. However, the domain decomposition re-introduces some sort of meshing which is not attractive. This paper formulates a simple meshfree solution procedure that overcomes the difficulty of having to solve fully populated large matrices through the local collocation technique. Two variants of the method are described, the direct and the indirect one.

2 Governing equations

Let us limit our discussion to solution of the general transport equation, defined on a fixed domain Ω with boundary Γ , standing for a reasonably broad spectrum of mass, energy, momentum, and species transfer problems

$$\frac{\partial}{\partial t} [\rho \mathfrak{A}(\Phi)] + \nabla \cdot [\rho \mathbf{v} \mathfrak{A}(\Phi)] = \nabla \cdot (\mathbf{D} \nabla \Phi) + S \quad (1)$$

with $\rho, \Phi, t, \mathbf{v}, \mathbf{D}$, and S standing for density, transport variable, time, velocity, diffusion matrix and source, respectively. The scalar function \mathfrak{A} stands for possible more involved constitutive relations between the conserved $\mathfrak{A}(\Phi)$ and the diffused Φ quantities (such as for example the relation between the enthalpy and the temperature). The solution of the governing equation for the transport variable at the final time $t_0 + \Delta t$ is sought, where t_0 represents the initial time and Δt the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial value of the transport variable



$\Phi(\mathbf{p}, t)$ at a point with position vector \mathbf{p} and time t_0 is defined through the known function Φ_0

$$\Phi(\mathbf{p}, t) = \Phi_0(\mathbf{p}, t); \mathbf{p} \in \Omega + \Gamma \quad (2)$$

The boundary Γ is divided into not necessarily connected parts $\Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^R$ with Dirichlet, Neumann and Robin type boundary conditions, respectively. At the boundary point \mathbf{p} with normal \mathbf{n}_Γ and time $t_0 \leq t \leq t_0 + \Delta t$, these boundary conditions are defined through known functions Φ_Γ^D , Φ_Γ^N , Φ_Γ^R , $\Phi_{\Gamma^{ref}}^R$

$$\Phi = \Phi_\Gamma^D; \mathbf{p} \in \Gamma^D, \quad \frac{\partial}{\partial n_\Gamma} \Phi = \Phi_\Gamma^N; \mathbf{p} \in \Gamma^N, \quad \frac{\partial}{\partial n_\Gamma} \Phi = \Phi_\Gamma^R (\Phi - \Phi_{\Gamma^{ref}}^R); \mathbf{p} \in \Gamma^R \quad (3,4,5)$$

3 Solution procedure

The representation of function over a set I of (in general) non-equally spaced ${}_I N$ nodes ${}_I \mathbf{p}_n$; $n = 1, 2, \dots, {}_I N$ is made in the following way

$$\Phi(\mathbf{p}) \approx \sum_{k=1}^{{}_I K} {}_I \psi_k(\mathbf{p}) {}_I \alpha_k \quad (6)$$

${}_I \psi_k$ stands for the shape functions, ${}_I \alpha_k$ for the coefficients of the shape functions, and ${}_I K$ represents the number of the shape functions. The left lower index on entries of expression (6) represents the sub-domain ${}_I \omega$ on which the coefficients ${}_I \alpha_k$ are determined. The sub-domains ${}_I \omega$ can in general be contiguous (overlapping) or non-contiguous (non-overlapping). Each of the sub-domains ${}_I \omega$ includes ${}_I N$ grid-points of which ${}_I N_\Omega$ are in the domain and ${}_I N_\Gamma$ are on the boundary. The coefficients can be calculated from the sub-domain nodes in two distinct ways. The first way is collocation (interpolation) and the second way is approximation by the least squares method. Only the simpler collocation version for calculation of the coefficients is considered in this text. Let us assume the known function values ${}_I \Phi_n$ in the nodes ${}_I \mathbf{p}_n$ of the subdomain ${}_I \omega$. The collocation implies

$$\Phi({}_I \mathbf{p}_n) = \sum_{k=1}^{{}_I N} {}_I \psi_k({}_I \mathbf{p}_n) {}_I \alpha_k \quad (7)$$

For the coefficients to be computable, the number of the shape functions has to match the number of the collocation points ${}_I K = {}_I N$, and the collocation



matrix has to be non-singular. The system of equations (7) can be written in a matrix-vector notation

$${}_I \underline{\Psi} {}_I \boldsymbol{\alpha} = {}_I \boldsymbol{\Phi}; \quad {}_I \underline{\Psi}_{kn} = {}_I \psi_k({}_I \mathbf{p}_n), \quad {}_I \boldsymbol{\Phi}_n = \Phi({}_I \mathbf{p}_n) \quad (8)$$

The coefficients ${}_I \boldsymbol{\alpha}$ can be computed by inverting the system (8)

$${}_I \boldsymbol{\alpha} = {}_I \underline{\Psi}^{-1} {}_I \boldsymbol{\Phi} \quad (9)$$

By taking into account the expressions for the calculation of the coefficients ${}_I \boldsymbol{\alpha}$, the collocation representation of function $\Phi(\mathbf{p})$ on subdomain ${}_I \omega$ can be expressed as

$$\Phi(\mathbf{p}) \approx \sum_{k=1}^N {}_I \psi_k(\mathbf{p}) \sum_{n=1}^N {}_I \underline{\Psi}_{kn}^{-1} {}_I \boldsymbol{\Phi}_n \quad (10)$$

Let us introduce the Cartesian coordinate system with base vectors $\mathbf{i}_\zeta; \zeta = x, y, z$ and coordinates $p_\zeta; \zeta = x, y, z$, i.e. $\mathbf{p} = \mathbf{i}_\zeta p_\zeta; \zeta = x, y, z$. The first partial spatial derivatives of $\Phi(\mathbf{p})$ on subdomain ${}_I \omega$ can be expressed as

$$\frac{\partial}{\partial p_\zeta} \Phi(\mathbf{p}) \approx \sum_{k=1}^N \frac{\partial}{\partial p_\zeta} {}_I \psi_k(\mathbf{p}) \sum_{n=1}^N {}_I \underline{\Psi}_{kn}^{-1} {}_I \boldsymbol{\Phi}_n; \quad \zeta = x, y, z \quad (11)$$

The second partial spatial derivatives of $\Phi(\mathbf{p})$ on subdomain ${}_I \omega$ can be expressed as

$$\frac{\partial^2}{\partial p_\zeta \partial p_\xi} \Phi(\mathbf{p}) \approx \sum_{k=1}^N \frac{\partial^2}{\partial p_\zeta \partial p_\xi} {}_I \psi_k(\mathbf{p}) \sum_{n=1}^N {}_I \underline{\Psi}_{kn}^{-1} {}_I \boldsymbol{\Phi}_n; \quad \zeta, \xi = x, y, z \quad (12)$$

3.1 The direct approach

The radial basis functions, such as multiquadrics, can be used for the shape function

$${}_I \psi_k(\mathbf{p}) = [{}_I r_k^2 + c^2]^{1/2}; \quad {}_I r_k^2 = (\mathbf{p} - {}_I \mathbf{p}_k) \cdot (\mathbf{p} - {}_I \mathbf{p}_k) \quad (13)$$

where c represents the shape parameter. The explicit values of the first and the second derivatives of $\psi_k(\mathbf{p})$ (in 2D, because of the space limitations) are

$$\frac{\partial}{\partial p_x} {}_I \psi_k(\mathbf{p}) = \frac{p_x - {}_I p_{kx}}{({}_I r_k^2 + c^2)^{1/2}}, \quad \frac{\partial}{\partial p_y} {}_I \psi_k(\mathbf{p}) = \frac{p_y - {}_I p_{ky}}{({}_I r_k^2 + c^2)^{1/2}} \quad (14,15)$$



$$\frac{\partial^2}{\partial p_x p_x} {}_i\psi_k(\mathbf{p}) = \frac{(p_y - {}_i p_{ky})^2 + c^2}{({}_i r_k^2 + c^2)^{3/2}}, \quad \frac{\partial^2}{\partial p_y p_y} {}_i\psi_k(\mathbf{p}) = \frac{(p_x - {}_i p_{kx})^2 + c^2}{({}_i r_k^2 + c^2)^{3/2}} \quad (16,17)$$

$$\frac{\partial^2}{\partial p_x p_y} {}_i\psi_k(\mathbf{p}) = \frac{\partial^2}{\partial p_y p_x} {}_i\psi_k(\mathbf{p}) = -\frac{(p_x - {}_i p_{kx})(p_y - {}_i p_{ky})}{({}_i r_k^2 + c^2)^{3/2}} \quad (18)$$

3.2 The indirect approach

In the indirect approach, the formulation of the problem starts with the representation of the second derivatives of the function with RBFs. The derivative expression is then integrated to yield an expression for the original function.

$$\frac{\partial^2}{\partial p_\xi p_\xi} {}_i\psi_k^{\xi\xi}(\mathbf{p}) = {}_i\psi_k(\mathbf{p}) = ({}_i r_k^2 + c^2)^{1/2}, \quad \frac{\partial}{\partial p_\xi} {}_i\psi_k^{\xi\xi}(\mathbf{p}) = \int {}_i\psi_k(\mathbf{p}) dp_\xi \quad (19,20)$$

$${}_i\psi_k^{\xi\xi}(\mathbf{p}) = \int \left(\int {}_i\psi_k(\mathbf{p}) dp_\xi \right) dp_\xi \quad (21)$$

The explicit values of the involved integrals are for the case with multiquadrics (in 2D, because of the space limitations)

$$\int {}_i\psi_k(\mathbf{p}) dp_x = \frac{(p_x - {}_i p_{kx})({}_i r_k^2 + c^2)^{1/2}}{2} + \frac{(p_y - {}_i p_{ky})^2 + c^2}{2} \ln \left[({}_i r_k^2 + c^2)^{1/2} + (p_x - {}_i p_{kx}) \right] \quad (22)$$

$$\int {}_i\psi_k(\mathbf{p}) dp_y = \frac{(p_y - {}_i p_{ky})({}_i r_k^2 + c^2)^{1/2}}{2} + \frac{(p_x - {}_i p_{kx})^2 + c^2}{2} \ln \left[({}_i r_k^2 + c^2)^{1/2} + (p_y - {}_i p_{ky}) \right] \quad (23)$$

$$\int \left(\int {}_i\psi_k(\mathbf{p}) dp_x \right) dp_x = \frac{(p_x - {}_i p_{kx})^2 - 2(p_y - {}_i p_{ky})^2 - 2c^2}{6} ({}_i r_k^2 + c^2)^{1/2} + \frac{(p_x - {}_i p_{kx}) \left[(p_y - {}_i p_{ky})^2 + c^2 \right]}{2} \ln \left[({}_i r_k^2 + c^2)^{1/2} + (p_x - {}_i p_{kx}) \right] \quad (24)$$

$$\int \left(\int {}_i\psi_k(\mathbf{p}) dp_y \right) dp_y = \frac{(p_y - {}_i p_{ky})^2 - 2(p_x - {}_i p_{kx})^2 - 2c^2}{6} ({}_i r_k^2 + c^2)^{1/2} + \frac{(p_y - {}_i p_{ky}) \left[(p_x - {}_i p_{kx})^2 + c^2 \right]}{2} \ln \left[({}_i r_k^2 + c^2)^{1/2} + (p_y - {}_i p_{ky}) \right] \quad (25)$$



$$\begin{aligned}
 \int \left(\int {}_l \psi_k(\mathbf{p}) dp_x \right) dp_y &= \int \left(\int {}_l \psi_k(\mathbf{p}) dp_y \right) dp_x = \\
 &-\frac{(p_y - {}_l p_{ky})}{18} \left[(p_y - {}_l p_{ky})^2 + 6c^2 - 6(p_x - {}_l p_{kx})({}_l r_k^2 + c^2)^{1/2} \right] \\
 &+ \frac{c^3}{6} \left[\arctan \frac{p_y - {}_l p_{ky}}{2} - \arctan \frac{(p_x - {}_l p_{kx})(p_y - {}_l p_{ky})}{c^2({}_l r_k^2 + c^2)^{1/2}} \right] \quad (26) \\
 &+ \frac{1}{6}(p_y - {}_l p_{ky}) \left[(p_y - {}_l p_{ky})^2 + 3c^2 \right] \ln \left[({}_l r_k^2 + c^2)^{1/2} + (p_x - {}_l p_{kx}) \right] \\
 &+ \frac{1}{6}(p_x - {}_l p_{kx}) \left[(p_x - {}_l p_{kx})^2 + 3c^2 \right] \ln \left[({}_l r_k^2 + c^2)^{1/2} + (p_y - {}_l p_{ky}) \right]
 \end{aligned}$$

The collocation in indirect approach implies

$$\Phi({}_l \mathbf{p}_n) = \sum_{k=1}^N {}_l \psi_k^{\xi\zeta}({}_l \mathbf{p}_n) {}_l \alpha_k^{\xi\zeta} \quad (27)$$

This can be performed in 6 different ways, depending on the choice of ξ and ζ , i.e. $\xi = \zeta = x$, $\xi = \zeta = y$, $\xi = \zeta = z$, $\xi = x, \zeta = y$ or $\xi = y, \zeta = x$, $\xi = x, \zeta = z$ or $\xi = z, \zeta = x$, $\xi = y, \zeta = z$ or $\xi = z, \zeta = y$. The system of equations (27) can be written in a matrix vector notation

$${}_l \underline{\Psi}^{\xi\zeta} {}_l \boldsymbol{\alpha}^{\xi\zeta} = {}_l \boldsymbol{\Phi}; \quad {}_l \psi_{kn}^{\xi\zeta} = {}_l \psi_k^{\xi\zeta}({}_l \mathbf{p}_n), \quad {}_l \Phi_n = \Phi({}_l \mathbf{p}_n) \quad (28)$$

The coefficients ${}_l \boldsymbol{\alpha}^{\xi\zeta}$ can be computed by inverting the system (28)

$${}_l \boldsymbol{\alpha}^{\xi\zeta} = {}_l \underline{\Psi}^{\xi\zeta-1} {}_l \boldsymbol{\Phi} \quad (29)$$

By taking into account the expressions for the calculation of the coefficients ${}_l \boldsymbol{\alpha}^{\xi\zeta}$ the indirect collocation representation of function $\Phi(\mathbf{p})$ on subdomain ${}_l \omega$ can be expressed as

$$\Phi(\mathbf{p}) \approx \sum_{k=1}^N {}_l \psi_k^{\xi\zeta}(\mathbf{p}) \sum_{n=1}^N {}_l \psi_{kn}^{\xi\zeta-1} {}_l \Phi_n \quad (30)$$

The first spatial partial derivatives of $\Phi(\mathbf{p})$ on subdomain ${}_l \omega$ can be expressed as

$$\frac{\partial}{\partial p_\zeta} \Phi(\mathbf{p}) \approx \sum_{k=1}^N \frac{\partial}{\partial p_\zeta} \psi_k^{\xi\zeta}(\mathbf{p}) \sum_{n=1}^N {}_l \psi_{kn}^{\xi\zeta-1} {}_l \Phi_n; \quad \zeta = x, y, z \quad (31)$$



The second spatial partial special derivatives of $\Phi(\mathbf{p})$ on subdomain ${}_l\omega$ can be expressed as

$$\frac{\partial^2}{\partial p_\zeta p_\xi} \Phi(\mathbf{p}) \approx \sum_{k=1}^{lN} {}_l\psi_k(\mathbf{p}) \sum_{n=1}^{lN} {}_l\psi_{kn}^{\xi\zeta-l} {}_l\Phi_n \quad ; \quad \zeta, \xi = x, y, z \quad (32)$$

3.3 Governing equation manipulations

What follows elaborates the semi-explicit solution of the general transport equation (1), subject to the initial condition (2), and the boundary conditions (3,4,5). The general transport equation can be transformed into the following expression by taking into account the explicit time discretisation

$$\frac{\rho \mathfrak{A}(\Phi) - \rho_0 \mathfrak{A}_0}{\Delta t} + \nabla \cdot [\rho_0 \mathbf{v}_0 \mathfrak{A}_0] = \nabla \cdot (\mathbf{D}_0 \nabla \Phi_0) + S \quad (33)$$

At time $t = t_0 + \Delta t$, the functions $\mathfrak{A}(\Phi)$ and $S(\Phi)$ can be expanded as

$$\mathfrak{A}(\Phi) \approx \bar{\mathfrak{A}} + \frac{\partial \bar{\mathfrak{A}}}{\partial \Phi} (\Phi - \bar{\Phi}), \quad S(\Phi) \approx \bar{S} + \frac{\partial \bar{S}}{\partial \Phi} (\Phi - \bar{\Phi}), \quad (34,35)$$

$$\frac{\rho \bar{\mathfrak{A}} + \rho \frac{\partial \bar{\mathfrak{A}}}{\partial \Phi} (\Phi - \bar{\Phi}) - \rho_0 \mathfrak{A}_0}{\Delta t} + \nabla \cdot [\rho_0 \mathbf{v}_0 \mathfrak{A}_0] = \nabla \cdot (\mathbf{D}_0 \nabla \Phi_0) + \bar{S} + \frac{\partial \bar{S}}{\partial \Phi} (\Phi - \bar{\Phi}) \quad (36)$$

Since the problem is non-linear, timestep iterations have to be employed. The over-bar denotes value from previous iteration. The unknown function value Φ_n in grid-point \mathbf{p}_n can be calculated as

$$\Phi_n = \frac{\rho_{n0} \mathfrak{A}_{n0} - \rho_n \bar{\mathfrak{A}}_n + \rho_n \frac{\partial \bar{\mathfrak{A}}}{\partial \Phi_n} \bar{\Phi}_n + \Delta t \left[\nabla \cdot (\mathbf{D}_{0n} \nabla \Phi_{0n}) - \nabla \cdot (\rho_{0n} \mathbf{v}_{0n} \mathfrak{A}_{0n}) + \bar{S}_n - \frac{\partial \bar{S}}{\partial \Phi_n} \bar{\Phi}_n \right]}{\rho \frac{\partial \bar{\mathfrak{A}}}{\partial \Phi_n} - \Delta t \frac{\partial \bar{S}}{\partial \Phi_n}} \quad (37)$$

The calculation of the convective and diffusive terms use expressions (11,12) in the direct approach and (31,32) in the indirect approach, respectively. The complete solution procedure follows the below defined steps 1-5. **Step 1:** First, the initial conditions are set in the domain and boundary nodes and the derivatives required in the convective and diffusive terms are calculated from the known nodal values. **Step 2:** The equation (37) is used to calculate the new



values of the variable ${}_l\Phi_n$ at time $t_0 + \Delta t$ in the domain nodes. **Step 3:** What follows in the steps 3 and 4 defines variable ${}_l\Phi_n$ at time $t_0 + \Delta t$ in the Dirichlet, Neumann, and Robin boundary nodes. For this purpose, in the step 3, the coefficients ${}_l\alpha$ have to be determined from the new values in the domain and from the information on the boundary conditions. Let us introduce domain, Dirichlet, Neumann, and Robin boundary indicators for this purpose. These indicators are defined as

$$\Upsilon_{\Omega n} = \begin{cases} 1; \mathbf{p}_n \in \Omega \\ 0; \mathbf{p}_n \notin \Omega \end{cases}, \quad \Upsilon_{\Gamma n}^D = \begin{cases} 1; \mathbf{p}_n \in \Gamma^D \\ 0; \mathbf{p}_n \notin \Gamma^D \end{cases}, \quad \Upsilon_{\Gamma n}^N = \begin{cases} 1; \mathbf{p}_n \in \Gamma^N \\ 0; \mathbf{p}_n \notin \Gamma^N \end{cases}, \quad \Upsilon_{\Gamma n}^R = \begin{cases} 1; \mathbf{p}_n \in \Gamma^R \\ 0; \mathbf{p}_n \notin \Gamma^R \end{cases} \quad (38)$$

The coefficients ${}_l\alpha$ are in the direct approach calculated from the system of equations

$$\begin{aligned} & \sum_{k=1}^N {}_l\Upsilon_{\Omega n} {}_l\psi_k({}_l\mathbf{p}_n) {}_l\alpha_k + \sum_{k=1}^N {}_l\Upsilon_{\Gamma n}^D {}_l\psi_k({}_l\mathbf{p}_n) {}_l\alpha_k \\ & + \sum_{k=1}^N {}_l\Upsilon_{\Gamma n}^N \frac{\partial}{{}_l\partial n_{\Gamma}} {}_l\psi_k({}_l\mathbf{p}_n) {}_l\alpha_k + \sum_{k=1}^N {}_l\Upsilon_{\Gamma n}^R \frac{\partial}{{}_l\partial n_{\Gamma}} {}_l\psi_k({}_l\mathbf{p}_n) {}_l\alpha_k = \quad (39) \\ & = {}_l\Upsilon_{\Omega n} {}_l\Phi_n + {}_l\Upsilon_{\Gamma n}^D {}_l\Phi_n^D + {}_l\Upsilon_{\Gamma n}^N {}_l\Phi_n^N + {}_l\Upsilon_{\Gamma n}^R {}_l\Phi_n^R \left(\sum_{k=1}^N {}_l\psi_k({}_l\mathbf{p}_n) {}_l\alpha_k - {}_l\Phi_{\Gamma ref n}^R \right) \end{aligned}$$

The calculation of the coefficients ${}_l\alpha$ follows from the following system of linear equations in the direct approach

$${}_l\Psi {}_l\alpha = {}_l\mathbf{b} \quad (40)$$

with the following explicit form of the system matrix coefficients

$$\begin{aligned} {}_l\Psi_{nk} = & {}_l\Upsilon_{\Omega n} {}_l\psi_k({}_l\mathbf{p}_n) + {}_l\Upsilon_{\Gamma n}^D {}_l\psi_k({}_l\mathbf{p}_n) \\ & + {}_l\Upsilon_{\Gamma n}^N \frac{\partial}{{}_l\partial n_{\Gamma}} {}_l\psi_k({}_l\mathbf{p}_n) + {}_l\Upsilon_{\Gamma n}^R \left[\frac{\partial}{{}_l\partial n_{\Gamma}} {}_l\psi_k({}_l\mathbf{p}_n) - {}_l\Phi_{\Gamma n}^R \sum_{k=1}^N {}_l\psi_k({}_l\mathbf{p}_n) \right] \quad (41) \end{aligned}$$

and with the following explicit form of the augmented right hand side vector

$${}_l\mathbf{b}_n = {}_l\Upsilon_{\Omega n} {}_l\Phi_n + {}_l\Upsilon_{\Gamma n}^D {}_l\Phi_n^D + {}_l\Upsilon_{\Gamma n}^N {}_l\Phi_n^N - {}_l\Upsilon_{\Gamma n}^R {}_l\Phi_{\Gamma n}^R {}_l\Phi_{\Gamma ref n}^R \quad (42)$$

The coefficients ${}_l\alpha^{\xi\xi}$ are in the indirect approach calculated from the system of equations

$$\begin{aligned}
& \sum_{k=1}^{lN} {}_l\Upsilon_{\Omega n} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) {}_l\alpha_k^{\xi\xi} + \sum_{k=1}^{lN} {}_l\Upsilon_{\Gamma n}^D {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) {}_l\alpha_k^{\xi\xi} \\
& + \sum_{k=1}^{lN} {}_l\Upsilon_{\Gamma n}^N \frac{\partial}{{\partial n}_{\Gamma}} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) {}_l\alpha_k^{\xi\xi} + \sum_{k=1}^{lN} {}_l\Upsilon_{\Gamma n}^R \frac{\partial}{{\partial n}_{\Gamma}} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) {}_l\alpha_k^{\xi\xi} = \\
& = {}_l\Upsilon_{\Omega n} {}_l\Phi_n + {}_l\Upsilon_{\Gamma n}^D {}_l\Phi_n^D + {}_l\Upsilon_{\Gamma n}^N {}_l\Phi_n^N + {}_l\Upsilon_{\Gamma n}^R {}_l\Phi_n^R \left(\sum_{k=1}^{lN} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) {}_l\alpha_k^{\xi\xi} - {}_l\Phi_{\Gamma ref n}^R \right)
\end{aligned} \quad (43)$$

The calculation of the coefficients ${}_l\alpha^{\xi\xi}$ follows from the following system of linear equations in the indirect approach

$${}_l\underline{\Psi}^{\xi\xi} {}_l\alpha^{\xi\xi} = {}_l\mathbf{b} \quad (44)$$

with the following explicit form of the system matrix coefficients

$$\begin{aligned}
{}_l\underline{\Psi}_{nk}^{\xi\xi} = & {}_l\Upsilon_{\Omega n} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) + {}_l\Upsilon_{\Gamma n}^D {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) \\
& + {}_l\Upsilon_{\Gamma n}^N \frac{\partial}{{\partial n}_{\Gamma}} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) + {}_l\Upsilon_{\Gamma n}^R \left[\frac{\partial}{{\partial n}_{\Gamma}} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) - {}_l\Phi_{\Gamma n}^R \sum_{k=1}^{lN} {}_l\psi_k^{\xi\xi}({}_l\mathbf{p}_n) \right]
\end{aligned} \quad (45)$$

and with the augmented right hand side vector equal to the one in the direct approach. The following coefficients need to be calculated in the general three-dimensional case in the indirect approach ${}_l\alpha^{xx}$, ${}_l\alpha^{yy}$, ${}_l\alpha^{zz}$, ${}_l\alpha^{xy}$ or ${}_l\alpha^{yx}$, ${}_l\alpha^{xy}$ or ${}_l\alpha^{yx}$, ${}_l\alpha^{xz}$ or ${}_l\alpha^{zx}$, ${}_l\alpha^{yz}$ or ${}_l\alpha^{zy}$. The indirect approach is thus computationally more demanding (6 systems of equations have to be solved in contrast to only one in direct approach), however it gives more accurate results as reported in [9]. **Step 4:** The unknown boundary values are set from equation (10) in the direct approach and from the equation (40) in the indirect approach. **Step 5:** The iterations over one time-step are completed when the criterion (46) is satisfied in all computational nodes $n = 1, 2, \dots, N$

$$\max |\Phi_n - \bar{\Phi}_n| \leq \Phi_{it} \quad (46)$$

The steady-state is achieved when the criterion (47) is satisfied in all computational nodes $n = 1, 2, \dots, N$

$$\max |\Phi_n - \Phi_0| \leq \Phi_{ste} \quad (47)$$

The parameters Φ_{it} and Φ_{ste} are defined as the iteration and the steady-state convergence margins. In case the iteration criterion is passed, calculation of the new time-step is performed. In case the steady-state criterion is passed or the



time of calculation exceeds the foreseen time of interest, the calculation is stopped.

4 Conclusions

This paper represents a new meshfree formulation for solving a wide range of transport phenomena. The governing equation is solved in its strong form. The developments are almost independent on the problem dimension. The complicated geometry is easy to cope with. No polygonisation is needed. No integrations are needed. The method appears efficient, because it does not require a solution of the large systems of equations like the original Kansa method. Instead, small systems of linear equations have to be solved in each timestep for each node and associated sub-domain, representing the most natural and automatic domain decomposition. The method is simple to learn and simple to code. The method can cope with large problems. A detailed comparative numerical study of both approaches will appear in one of our future publications.

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References

- [1] Atluri, S.N. & Shen, S. *The Meshless Method*, Tech Science Press, Forsyth, 2002.
- [2] Liu, G.R. *Mesh Free Methods*, CRC Press, Boca Raton, 2003.
- [3] Atluri, S.N. *The Meshless Method for Domain and BIE Discretisations*, Tech Science Press, Forsyth, 2004.
- [4] Šarler, B. *Chapter 9: Meshless Methods*, in: Nowak, A.J. (Ed.) *Advanced Numerical Methods in Heat Transfer*, Silesian Technical University Press, Poland, 2004.
- [5] Buhmann, M.D. *Radial Basis Function: Theory and Implementations*, Cambridge University Press, Cambridge, 2003.
- [6] Kansa, E.J. Multiquadrics – a scattered data approximation scheme with applications to computational fluid dynamics-II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, *Computers and Mathematics with Application*, 1990, **19**, 147-161.
- [7] Power, H. & Barraco, W. A Comparison analysis between unsymmetric and symmetric RBFs for the numerical solution of PDF's, *Computers and Mathematics with Applications*, 2002, **43**, 551-583.
- [8] Chen, W. New RBF collocation schemes and kernel RBFs with applications, *Lecture Notes in Computational Science and Engineering*, 2002, **26**, 75-86.



- [9] Mai-Duy, N. & Tran-Cong, T. Indirect RBFN method with thin plate splines for numerical solution of differential equations, *Computer Modeling in Engineering & Sciences*, 2003, **4**, 85-102.
- [10] Šarler, B. A radial basis function collocation approach in computational fluid dynamics, *Computer Modeling in Engineering & Sciences*, 2005, (in print).
- [11] Mai-Duy, N & Tran-Cong, T. Numerical solution of Navier-Stokes equations using multiquadric radial basis function networks, *Neural Networks*, 2001, **14**, 185-199.
- [12] Šarler, B.; Perko, J. & Chen, C.S. Radial basis function collocation method solution of natural convection in porous media, *Int.J.Numer.Methods Heat & Fluid Flow*, 2004, **14**, 187-212.
- [13] Kovačević, I.; Poredoš, A. & Šarler, B. (2003): Solving the Stefan problem by the RBFCM, *Numer.Heat Transfer, Part B: Fundamentals*, 2003, **44**, 1-24.
- [14] Mai-Duy, N. & Tran-Cong, T. Mesh-free radial basis function network methods with domain decomposition for approximation of functions and numerical solution of Poisson's equations, *Engineering Analysis with Boundary Elements*, (2002), **26**, 133-156.

